On the Complexity of the Equational Theory of Relational Action Algebras

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Abstract. Pratt [22] defines action algebras as Kleene algebras with residuals. In [9] it is shown that the equational theory of *-continuous action algebras (lattices) is Π_1^0−complete. Here we show that the equational theory of relational action algebras (lattices) is Π_1^0−hard, and some its fragments are Π_1^0−complete. We also show that the equational theory of action algebras (lattices) of regular languages is Π_1^0−complete.

1 Introduction

A Kleene algebra is an algebra $\mathcal{A} = (A, \vee, \cdot, ^*, 0, 1)$ such that $(A, \vee, 0)$ is a join semilattice with the least element 0, $(A, \cdot, 1)$ is a monoid, product $\cdot$ distributes over join $\vee$, 0 is an annihilator for product, and * is a unary operation on A, fulfilling the conditions:

$$1 \vee aa^* \leq a^* \ , \ 1 \vee a^*a \leq a^* \ , \tag{1}$$

$$ab \leq b \Rightarrow a^*b \leq b \ , \ ba \leq b \Rightarrow ba^* \leq b \ , \tag{2}$$

for all $a, b \in A$. One defines: $a \leq b$ iff $a \vee b = b$. The notion of a Kleene algebra has been introduced by Kozen [15, 16] to provide an algebraic axiomatization of the algebra of regular expressions. Regular expressions on an alphabet Σ can be defined as terms of the first-order language of Kleene algebras whose variables are replaced by symbols from Σ (treated as individual constants). Each regular expression α on Σ denotes a regular language $L(\alpha) \subseteq \Sigma^*$. The Kozen completeness theorem states that $L(\alpha) = L(\beta)$ if and only if $\alpha = \beta$ is valid in Kleene algebras.

The class of Kleene algebras is a quasi-variety, but not a variety. Redko [23] shows that the equations true for regular expressions cannot be axiomatized by any finite set of equations. Pratt [22] shows that the situation is different for Kleene algebras with residuals, called action algebras. *An action algebra* is a Kleene algebra $\mathcal{A}$ supplied with two binary operations $/, \backslash$, fulfilling the equivalences:

$$ab \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a\backslash c \ , \tag{3}$$

for all $a, b, c \in A$. Operations $/, \backslash$ are called *the left* and *right residual*, respectively, with respect to product. Pratt writes $a \rightarrow b$ for $a\backslash b$ and $a \leftarrow b$ for a/b; we use the slash notation of Lambek [18]. Pratt [22] proves that the class of action algebras is a finitely based variety. Furthermore, in the language without residuals, the equations true in all action algebras are the same as those true in all Kleene algebras. Consequently, in the language with residuals, one obtains a finite, equational axiomatization of the algebra of regular expressions.

On the other hand, the logic of action algebras differs in many essential aspects from the logic of Kleene algebras. Although regular languages are (effectively) closed under residuals, the Kozen completeness theorem is not true for terms with residuals. For instance, since $L(a) = \{a\}$, then $L(a/a) = \{\epsilon\}$, while $a/a = 1$ is not true in action algebras (one only gets $1 \leq a/a$). It is known that $L(\alpha) = L(\beta)$ iff $\alpha = \beta$ is valid in relational algebras (α, β do not contain residuals). Consequently, the equational theory of Kleene algebras equals the equational theory of relational Kleene algebras. This is not true for action algebras (see below).

A Kleene algebra is said to be **-continuous*, if $xa^*y = \sup\{xa^ny : n \in \omega\}$, for all elements x, a, y. Relational algebras with operations defined in the standard way and algebras of (regular) languages are *-continuous. The equational theory of Kleene algebras equals the equational theory of *-continuous Kleene algebras. Again, it is not the case for action algebras. The equational theory of all action algebras is recursively enumerable (it is not known if it is decidable), while the equational theory of *-continuous action algebras is Π^0_1-complete [9], and consequently, it possesses no recursive axiomatization.

In this paper we study the complexity of relational action algebras and lattices. *An action lattice* is an action algebra $\mathcal{A}$ supplied with meet $\wedge$ such that $(A, \vee, \wedge)$ is a lattice; Kleene lattices are defined in a similar way. If $\mathcal{K}$ is a class of algebras, then $\mathrm{Eq}(\mathcal{K})$ denotes the equational theory of $\mathcal{K}$, this means, the set of all equations valid in $\mathcal{K}$. KA, KL, ACTA, ACTL will denote the classes of Kleene algebras, Kleene lattices, action algebras, and action lattices, respectively. KA* denotes the class of *-continuous Kleene algebras, and similarly for the other classes.

Let U be a set. $P(U^2)$ (the powerset of U^2) is the set of all binary relations on U. For $R, S \subseteq U^2$, one defines: $R \vee S = R \cup S$, $R \wedge S = R \cap S$, $R \cdot S = R \circ S$, $1 = I_U = \{(x, x) : x \in U\}$, $0 = \emptyset$, $R^0 = I_U$, $R^{n+1} = R^n \circ R$, $R^* = \bigcup_{n \in \omega} R^n$, and:

$$R/S = \{(x, y) \in U^2 : \{(x, y)\} \circ S \subseteq R\} \ , \tag{4}$$

$$R\backslash S = \{(x, y) \in U^2 : R \circ \{(x, y)\} \subseteq S\} \ , \tag{5}$$

.

$P(U^2)$ with so-defined operations and designated elements is an action lattice (it is a complete lattice). Algebras of this form will be called *relational action lattices*; without meet, they will be called *relational action algebras*. Omitting residuals, one gets relational Kleene lattices and algebras, respectively. RKA, RKL, RACTA, RACTL will denote the classes of relational Kleene algebras, relational Kleene lattices, relational action algebras and relational action lattices, respectively.

All relational algebras and lattices, mentioned above, are *-continuous. Consequently, Eq(KA)⊆Eq(KA*)⊆Eq(RKA), and similar inclusions are true for classes KL, KL*, RKL, classes ACTA, ACTA*, RACTA, and classes ACTL, ACTL*, RACTL. It is known that Eq(KA)=Eq(KA*)=Eq(RKA) (this follows from the Kozen completeness theorem and the fact mentioned in the third paragraph of this section). We do not know if Eq(KL)=Eq(KL*). All relational Kleene lattices are distributive lattices, but there exist nondistributive *-continuous Kleene lattices, which yields Eq(KL*)≠Eq(RKL). Since Eq(ACTA) is Σ^0_1, and Eq(ACTA*) is Π^0_1-complete, then Eq(ACTA)≠Eq(ACTA*); also, Eq(ACTL)≠Eq(ACTL*), for similar reasons [9].

It is easy to show that Eq(ACTA*) (resp. Eq(ACTL*)) is strictly contained in Eq(RACTA) (resp. Eq(RACTL)); see section 2. Then, Π^0_1-completeness of the former theory does not directly provide any information on the complexity of the latter. In section 3, we prove that Eq(RACTA) and Eq(RACTL) are Π^0_1-hard. The argument is similar to that in [9] which yields Π^0_1-hardness of Eq(ACTA*) and Eq(ACTL*): we show that the total language problem for context-free grammars is reducible to Eq(RACTL) and Eq(RACTA).

We do not know whether Eq(RACTA) and Eq(RACTL) are Π^0_1. In [21], it has been shown that Eq(ACTA*) and Eq(ACTL*) are Π^0_1, using an infinitary logic for ACTL* [9], which satisfies the cut-elimination theorem and a theorem on elimination of negative occurrences of *. The elimination procedure replaces each negative occurrence of α^* by the disjunction $1 \vee \alpha \vee \ldots \vee \alpha^n$, for some $n \in \omega$. As a result, one obtains expressions which contain 1. Unfortunately, the exact complexity of Eq(RACTA), Eq(RACTL) is not known; without * they are Σ^0_1. Andréka and Mikulás [1] prove a representation theorem for residuated meet semilattices which implies that, in language with $\wedge, \cdot, /, \backslash$ only, order formulas $\alpha \leq \beta$ valid in RACTL possess a cut-free, finitary axiomatization (the Lambek calculus admitting meet and empty antecedents of sequents), and consequently, the validity problem for such formulas is decidable (other results and proofs can be found in [10, 8]). We use this fact in section 3 (lemma 2) to prove the results mentioned in the above paragraph and to prove Π^0_1-completeness of some fragments of Eq(RACTL).

In section 4, we consider analogous questions for the equational theory of action algebras (lattices) of regular languages, and we show that this theory is Π^0_1-complete. We use the completeness of the product-free fragment L (with $\wedge$) with respect to algebras of regular languages; the proof is a modification of the proof of finite model property for this system [5, 7].

Our results show that there exists no finitary dynamic logic (like PDL), complete with respect to standard relational frames, which handles programs formed by residuals and regular operations. Programs with residuals can express the weakest prespecification and postspecification of a program and related conditions; see Hoare and Jifeng [13].

2 Sequent systems

To provide a cut-free axiom system for the logic of *-continuous action algebras (lattices) it is expedient to consider *sequents* of the form $\Gamma \Rightarrow \alpha$ such that Γ is a finite sequence of terms (of the first-order language of these algebras), and α is a term. (Terms are often called *formulas.*) Given an algebra $\mathcal{A}$, *an assignment* is a homomorphism f from the term algebra to $\mathcal{A}$; one defines $f(\Gamma)$ by setting: $f(\epsilon) = 1$, $f(\alpha_1, \dots, \alpha_n) = f(\alpha_1) \cdot \dots \cdot f(\alpha_n)$. One says that $\Gamma \Rightarrow \alpha$ is *true* in $\mathcal{A}$ under f, if $f(\Gamma) \leq f(\alpha)$. Clearly, $f(\alpha) = f(\beta)$ iff both $f(\alpha) \leq f(\beta)$ and $f(\beta) \leq f(\alpha)$. A sequent is said to be *true* in $\mathcal{A}$, if it is true in $\mathcal{A}$ under any assignment, and *valid* in a class $\mathcal{K}$, if it is true in all algebras from $\mathcal{K}$. Since Eq($\mathcal{K}$) and the set of sequents valid in $\mathcal{K}$ are simply interpretable in each other, then the complexity of one of these sets equals the complexity of the other.

The sequents valid in ACTL* can be axiomatized by the following system. The axioms are:

$$\text{(I) } \alpha \Rightarrow \alpha \ \ , \text{(1) } \Rightarrow 1 \ \ , \text{(0) } \alpha, 0, \beta \Rightarrow \gamma \ \ , \tag{6}$$

and the inference rules are:

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma;\ \Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \gamma}\ \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}, \tag{7}$$

$$\frac{\Gamma, \alpha_i, \Delta \Rightarrow \gamma}{\Gamma, \alpha_1 \wedge \alpha_2, \Delta \Rightarrow \gamma},\ \frac{\Gamma \Rightarrow \alpha;\ \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}, \tag{8}$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \gamma},\ \frac{\Gamma \Rightarrow \alpha;\ \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}, \tag{9}$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma;\ \Phi \Rightarrow \beta}{\Gamma, \alpha/\beta, \Phi, \Delta \Rightarrow \gamma},\ \frac{\Gamma, \beta \Rightarrow \alpha}{\Gamma \Rightarrow \alpha/\beta}, \tag{10}$$

$$\frac{\Gamma, \beta, \Delta \Rightarrow \gamma;\ \Phi \Rightarrow \alpha}{\Gamma, \Phi, \alpha\backslash\beta, \Delta \Rightarrow \gamma},\ \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}, \tag{11}$$

$$\frac{\Gamma, \Delta \Rightarrow \alpha}{\Gamma, 1, \Delta \Rightarrow \alpha}, \tag{12}$$

$$\frac{(\Gamma, \alpha^n, \Delta \Rightarrow \beta)_{n \in \omega}}{\Gamma, \alpha^*, \Delta \Rightarrow \beta};\ \frac{\Gamma_1 \Rightarrow \alpha; \dots; \Gamma_n \Rightarrow \alpha}{\Gamma_1, \dots, \Gamma_n \Rightarrow \alpha^*}. \tag{13}$$

These rules are typical left- and right-introduction rules for Gentzen-style sequent systems. For each pair of rules, the left-hand rule will be denoted by (operation-L), and the right-hand rule by (operation-R). For instance, rules (7) will be denoted ($\vee$−L) and ($\vee$−R), respectively. Rule (12) will be denoted (1-L). Rule (*-L) is an infinitary rule (a kind of ω−rule); here α^n stands for the sequence of n copies of α. (*-R) denotes an infinite set of finitary rules: one for any fixed $n \in \omega$. For $n = 0$, (*-R) has the empty set of premises, so it is, actually, an axiom $\Rightarrow \alpha^*$; this yields $1 \Rightarrow \alpha^*$, by (1-L).

Without * and rules (13), the system is known as Full Lambek Calculus (FL); see Ono [19], Jipsen [14]. The rule (CUT):

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \beta;\ \Phi \Rightarrow \alpha}{\Gamma, \Phi, \Delta \Rightarrow \beta} \tag{14}$$

is admissible in FL, this means: if both premises are provable in FL, then the conclusion is provable in FL [19]. The $(\cdot, /, \backslash)-$fragment of FL is the Lambek calculus L (admitting empty antecedents of sequents), introduced by Lambek [18] (in a form not admitting empty antecedents) who has proved the cut-elimination theorem for L.

A residuated lattice is an algebra $\mathcal{A} = (A, \vee, \wedge, \cdot, /, \backslash, 0, 1)$ such that $(A, \vee, \wedge)$ is a lattice with the least element 0, $(A, \cdot, 1)$ is a monoid, and $/, \backslash$ are residuals for product (they fulfill (3)). It is known that FL is complete with respect to residuated lattices: a sequent is provable in FL iff it is valid in the class of residuated lattices. *A residuated monoid* is a structure $\mathcal{A} = (A, \leq, \cdot, /, \backslash, 1)$ such that $(A, \leq)$ is a poset, $(A, \cdot, 1)$ is a monoid, and $/, \backslash$ are residuals for product. L is complete with respect to residuated monoids. These completeness theorems can be proved in a standard way: soundness is obvious, and completeness can be shown by the construction of a Lindenbaum algebra. Residuated monoids and lattices are applied in different areas of logic and computer science; see e.g. [19, 20, 6].

The following monotonicity conditions are true in all residuated monoids: if $a \leq c$ and $b \leq d$, then $ab \leq cd$, $a/d \leq c/b$, $d\backslash a \leq b\backslash c$ (in lattices also: $a \vee b \leq c \vee d$, $a \wedge b \leq c \wedge d$, in action algebras also: $a^* \leq c^*$).

FL with * and rules (13) has been introduced in [9] and denoted by ACTω. The set of provable sequents can be defined in the following way. For a set X, of sequents, $C(X)$ is defined as the set of all sequents derivable from sequents from X by a single application of some inference rule (axioms are treated as inference rules with the empty set of premises). Then, $C(\emptyset)$ is the set of all axioms. One defines a transfinite chain C_ζ, for ordinals ζ, by setting: $C_0 = \emptyset$, $C_{\zeta+1} = C(C_\zeta)$, $C_\lambda = \bigcup_{\zeta<\lambda} C_\zeta$. Since C is a monotone operator and $C_0 \subseteq C_1$, then $C_\zeta \subseteq C_{\zeta+1}$, for all ζ, and consequently, $C_\zeta \subseteq C_\eta$ whenever $\zeta < \eta$. The join of this chain equals the set of sequents provable in ACTω. *The rank* of a provable sequent equals the least ζ such that this sequent belongs to C_ζ.

The cut-elimination theorem for ACTω is proved in [21] by a triple induction: (1) on the complexity of formula α in (CUT), (2) on the rank of $\Gamma, \alpha, \Delta \Rightarrow \beta$, (3) on the rank of $\Phi \Rightarrow \alpha$ (following an analogous proof for L in [4]). Let us show one case of induction (1): $\alpha = \gamma^*$. Assume that $\Gamma, \alpha, \Delta \Rightarrow \beta$ and $\Phi \Rightarrow \alpha$ are provable. We start induction (2). If the left premise is an axiom (I), then the conclusion of (CUT) is the right premise. If the left premise is an axiom (0), then the conclusion of (CUT) is also an axiom (0). Assume that the left premise gets its rank on the basis of an inference rule R; then, each premise of R is of a smaller rank. If R is any rule, not introducing the designated occurrence of α, then we directly apply the hypothesis of induction (2). If R introduces the designated occurrence of α, then R is (*-L) with premises $\Gamma, \gamma^n, \Delta \Rightarrow \beta$, for all

$n \in \omega$. We start induction (3). If $\Phi \Rightarrow \alpha$ is an axiom (I), then the conclusion of (CUT) is the left premise of (CUT). If $\Phi \Rightarrow \alpha$ is an axiom (0), then the conclusion of (CUT) is also an axiom (0). If $\Phi \Rightarrow \alpha$ is a conclusion of (*-R), then the premises are $\Phi_1 \Rightarrow \gamma, \ldots, \Phi_n \Rightarrow \gamma$, for some $n \in \omega$. For $n = 0$, we get $\Phi = \epsilon$, and the conclusion of (CUT) is the premise of (*-L) for $n = 0$. For $n > 0$, one of the premises of (*-L) is $\Gamma, \gamma^n, \Delta \Rightarrow \beta$, and we use n times the hypothesis of induction (1). If $\Phi \Rightarrow \alpha$ is a conclusion of a rule different from (*-R), then we directly apply the hypothesis of induction (3).

Since the rule (CUT) is admissible in ACTω, then a standard argument yields the completeness of ACTω with respect to *-continuous action lattices [21]. Soundness is obvious, and completeness can be shown by the construction of a Lindenbaum algebra. Using (1-L),(*-L) and (*-R), one easily proves $1 \Rightarrow \alpha^*$, $\alpha, \alpha^* \Rightarrow \alpha^*$ and, using (CUT), derives the following rules:

$$\frac{\alpha, \beta \Rightarrow \beta}{\alpha^*, \beta \Rightarrow \beta}, \ \frac{\beta, \alpha \Rightarrow \beta}{\beta, \alpha^* \Rightarrow \beta}, \tag{15}$$

and consequently, the Lindenbaum algebra is an action lattice. By (*-L), it is *-continuous.

Since ACTω is cut-free, then it possesses the subformula property: every provable sequent admits a proof in which all sequents consist of subformulas of formulas appearing in this sequent. In particular, ACTω is a conservative extension of all its fragments, obtained by a restriction of the language, e.g. L, FL, the $\vee-$free fragment, the $\wedge-$free fragment, and so on. All *-free fragments are finitary cut-free systems, admitting a standard proof-search decision procedure. So, they are decidable.

Now, we show that Eq(ACTA*)$\neq$Eq(RACTA). In relational algebras, for $R, S \subseteq I_U$, we have $R \circ S = R \cap S$. Fix a variable p. In L, from $p \Rightarrow p$, one infers $\Rightarrow p/p$, by (/-R). Then, $1 \Rightarrow 1$ yields $1/(p/p) \Rightarrow 1$, by (/-L). So, the sequent $1/(p/p) \Rightarrow (1/(p/p)) \cdot (1/(p/p))$ is valid in RACTA. It is not valid in ACTA*, since it is not provable in L. (Use the proof-search procedure; notice that $\Rightarrow p$, $p/p \Rightarrow 1$, $\Rightarrow 1/(p/p)$ are not provable.) The same example shows Eq(ACTL*)$\neq$Eq(RACTL) (another proof: the distribution of $\wedge$ over $\vee$ is not valid in ACTL*, since it is not provable in FL).

We define positive and negative occurrences of subterms in terms: α is positive in α; if γ is positive (resp. negative) in α or β, then it is positive (resp. negative) in $\alpha \vee \beta$, $\alpha \wedge \beta$, $\alpha \cdot \beta$, α^*; if γ is positive (resp. negative) in β, then it is positive (resp. negative) in β/α, $\alpha\backslash\beta$; if γ is positive (resp. negative) in α, then it is negative (resp. positive) in β/α, $\alpha\backslash\beta$.

For $n \in \omega$, let $\alpha^{\leq n}$ denote $\alpha^0 \vee \ldots \vee \alpha^n$; here α^i stands for the product of i copies of α and α^0 is the constant 1. We define two term transformations P_n, N_n, for any $n \in \omega$ [9]. Roughly, $P_n(\gamma)$ (resp. $N_n(\gamma)$) arises from γ by replacing any positive (resp. negative) subterm of the form α^* by $\alpha^{\leq n}$.

$$P_n(\alpha) = N_n(\alpha) = \alpha \ \text{, if } \alpha \text{ is a variable or a constant,} \tag{16}$$

$$P_n(\alpha \circ \beta) = P_n(\alpha) \circ P_n(\beta) \ \text{, for } \circ = \vee, \wedge, \cdot \ \text{,} \tag{17}$$

$$N_n(\alpha \circ \beta) = N_n(\alpha) \circ N_n(\beta) \ , \text{ for } \circ = \vee, \wedge, \cdot \ , \tag{18}$$

$$P_n(\alpha/\beta) = P_n(\alpha)/N_n(\beta) \ , P_n(\alpha\backslash\beta) = N_n(\alpha)\backslash P_n(\beta) \ , \tag{19}$$

$$N_n(\alpha/\beta) = N_n(\alpha)/P_n(\beta) \ , N_n(\alpha\backslash\beta) = P_n(\alpha)\backslash N_n(\beta) \ , \tag{20}$$

$$P_n(\alpha^*) = (P_n(\alpha))^{\leq n} \ , N_n(\alpha^*) = (N_n(\alpha))^* \ . \tag{21}$$

For a sequent $\Gamma \Rightarrow \alpha$, we set $N_n(\Gamma \Rightarrow \alpha) = P_n(\Gamma) \Rightarrow N_n(\alpha)$, where:

$$P_n(\epsilon) = \epsilon \ , P_n(\alpha_1, \ldots, \alpha_k) = P_n(\alpha_1), \ldots, P_n(\alpha_k) \ . \tag{22}$$

A term occurs positively (resp. negatively) in $\Gamma \Rightarrow \alpha$ if it occurs positively (resp. negatively) in α or negatively (resp. positively) in Γ.

Palka [21] proves the following theorem on elimination of negative occurrences of *: for any sequent $\Gamma \Rightarrow \alpha$, this sequent is provable in ACTω iff, for all $n \in \omega$, the sequent $N_n(\Gamma \Rightarrow \alpha)$ is provable in ACTω.

As a consequence of this theorem, the set of sequents provable in ACTω is Π^0_1. Indeed, the condition

$$N_n(\Gamma \Rightarrow \alpha) \text{ is provable in ACT}\omega \tag{23}$$

is recursive, since $N_n(\Gamma \Rightarrow \alpha)$ contains no negative occurrences of *, whence it is provable in ACTω iff it is provable in ACTω^-, i.e. ACTω without rule (*-L), and the latter system is finitary and admits an effective proof-search procedure. Actually, no result of the present paper relies upon Palka's theorem except for some remark at the end of section 3.

3 Eq(RACTL) and Eq(RACTA) are $\boldsymbol{\Pi^0_1}$−hard

A context-free grammar is a quadruple $G = (\Sigma, N, s, P)$ such that Σ, N are disjoint, finite alphabets, $s \in N$, and P is a finite set of production rules of the form $p \mapsto x$ such that $p \in N$, $x \in (\Sigma \cup N)^*$. Symbols in Σ are called *terminal* symbols and symbols in N are called *nonterminal* symbols. The relation $\Rightarrow_G$ is defined as follows: $x \Rightarrow_G y$ iff, for some $z, u, v \in (\Sigma \cup N)^*$, $p \in N$, we have $x = upv$, $y = uxv$ and $(p \mapsto x) \in P$. The relation $\Rightarrow^*_g$ is the reflexive and transitive closure of $\Rightarrow_G$. *The language* of G is the set:

$$L(G) = \{x \in \Sigma^* : s \Rightarrow_G x\} \ . \tag{24}$$

A context-free grammar G is said to be $\epsilon-$free, if $x \neq \epsilon$, for any rule $p \mapsto x$ in P. If G is $\epsilon-$free, then $\epsilon \notin L(G)$. The following problem is Π^0_1-complete [12]: for any context-free grammar G, decide if $L(G) = \Sigma^*$. Since the problem if $\epsilon \in L(G)$ is decidable, and every grammar G can be effectively transformed into an $\epsilon-$free grammar G' such that $L(G') = L(G) - \{\epsilon\}$, then also the following problem is Π^0_1-complete: for any $\epsilon-$free context-free grammar G, decide if $L(G) = \Sigma^+$ [9].

Types will be identified with (/)−terms of the language of ACTω, this means, terms formed out of variables by means of / only. *A Lambek categorial grammar* is a tuple $G = (\Sigma, I, s)$ such that Σ is a finite alphabet, I is a finite relation between symbols from Σ and types, and s is a designated variable. For $a \in \Sigma$, $I(a)$ denotes the set of all types α such that $aI\alpha$. (The relation I is called the initial type assignment of G.) For a string $a_1 \ldots a_n \in \Sigma^+$, $a_i \in \Sigma$, and a type α, we write $a_1 \ldots a_n \rightarrow_G \alpha$ if there are $\alpha_1 \in I(a_1)$, …, $\alpha_n \in I(a_n)$ such that $\alpha_1, \ldots, \alpha_n \Rightarrow \alpha$ is provable in L. We define the language of G as the set of all $x \in \Sigma^+$ such that $x \rightarrow_G s$. (Notice that we omit commas between symbols in strings on Σ, but we write them in sequences of terms appearing in sequents.) In general, Lambek categorial grammars admit types containing $\cdot, \backslash$ and, possibly, other operations [6], but we do not employ such grammars in this paper.

It is well-known that, for any $\epsilon-$free context-free grammar G, one can effectively construct a Lambek categorial grammar G' with the same alphabet Σ and such that $L(G) = L(G')$; furthermore, the relation I of G' employs very restricted types only: of the form p, p/q, $(p/q)/r$, where p, q, r are variables. This fact has been proved in [2] for classical categorial grammars and extended to Lambek categorial grammars by several authors; see e.g. [4, 9]. One uses the fact that, for sequents $\Gamma \Rightarrow s$ such that Γ is a finite sequence of types of the above form and s is a variable, Γ reduces to s in the sense of classical categorial grammars iff $\Gamma \Rightarrow s$ is provable in L.

Consequently, the problem if $L(G) = \Sigma^+$, for Lambek categorial grammars G, is Π^0_1-complete. In [9] it is shown that this problem is reducible to the decision problem for ACTω. Then, Eq(ACTL*) is Π^0_1-hard, and the same holds for Eq(ACTA*). Below we show that this reduction also yields the Π^0_1-hardness of Eq(RACTL) and Eq(RACTA).

Let $G = (\Sigma, I, s)$ be a Lambek categorial grammar. We can assume $I_G(a) \neq \emptyset$, for any $a \in \Sigma$; otherwise $L(G) \neq \Sigma^+$ immediately. We can also assume that all types involved in I are of one of the forms: p, p/q, $(p/q)/r$, where p, q, r are variables. Fix $\Sigma = \{a_1, \ldots, a_k\}$, where $a_i \neq a_j$ for $i \neq j$. Let $\alpha^i_1, \ldots, \alpha^i_{n_i}$ be all distinct types $\alpha \in I(a_i)$. For any $i = 1, \ldots, k$, we form a term $\beta_i = \alpha^i_1 \wedge \ldots \wedge \alpha^i_{n_i}$. We also define a term $\gamma(G) = \beta_1 \vee \ldots \vee \beta_k$. The following lemma has been proved in [9].

Lemma 1. *$L(G) = \Sigma^+$ iff $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is provable in ACTω.*

Proof. For the sake of completeness, we sketch the proof. $L(G) = \Sigma^+$ iff, for all $n \geq 1$ and all sequences $(i_1, \ldots, i_n)$ of integers from the set $[k] = \{1, \ldots, k\}$, $a_{i_1} \ldots a_{i_n} \rightarrow_G s$. The latter condition is equivalent to the following: for any $j = 1, \ldots, n$, there exists $\alpha^{i_j}_{l_j} \in I(a_{i_j})$ such that $\alpha^{i_1}_{l_1}, \ldots, \alpha^{i_n}_{l_n} \Rightarrow s$ is provable in L. The latter condition is equivalent to the following: $\beta_{i_1}, \ldots, \beta_{i_n} \Rightarrow s$ is provable in FL. One uses the following fact: if $\Gamma \Rightarrow \alpha$ is a $(\wedge, /)-$sequent in which all occurrences of $\wedge$ are negative, and $\gamma_1 \wedge \gamma_2$ occurs in this sequent (as a subterm of a term), then $\Gamma \Rightarrow \alpha$ is provable in FL iff both $\Gamma' \Rightarrow \alpha'$ and $\Gamma'' \Rightarrow \alpha''$ are provable in FL, where $\Gamma' \Rightarrow \alpha'$ (resp. $\Gamma'' \Rightarrow \alpha''$) arises from $\Gamma \Rightarrow \alpha$ by replacing the designated occurrence of $\gamma_1 \wedge \gamma_2$ by γ_1 (resp. γ_2). Now, for $n \geq 1$, $\beta_{i_1}, \ldots, \beta_{i_n} \Rightarrow s$ is provable in FL, for all sequents $(i_1, \ldots, i_n) \in [k]^n$, iff $(\gamma(G))^n \Rightarrow s$ is provable in FL (here we use the distribution of product over join). By (*-L), (*-R), the latter condition is equivalent to the following: $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is provable in ACTω. □

Andréka and Mikulás [1] prove that every residuated meet semilattice is embeddable into a relational algebra. The embedding h does not preserve 1; one only gets: $1 \leq a$ iff $I_U \subseteq h(a)$. It follows that the $(\wedge, \cdot, /, \backslash)$−fragment of FL is (even strongly) complete with respect to relational algebras, which is precisely stated by the following lemma (from [1]; also see [10, 8] for different proofs).

Lemma 2. *Let $\Gamma \Rightarrow \alpha$ be a $(\wedge, \cdot, /, \backslash)$−sequent of the language of FL. Then, $\Gamma \Rightarrow \alpha$ is provable in FL iff it is valid in RACTL.*

We use lemmas 1 and 2 to prove the following theorem.

Theorem 1. *Eq(RACTL) is Π^0_1−hard.*

Proof. We show that $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is provable in ACTω iff the sequent is valid in RACTL. The implication ($\Rightarrow$) is obvious. Assume that $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is not provable in ACTω. Then, for some $n \in \omega$, $(\gamma(G))^n, \gamma(G) \Rightarrow s$ is not provable in ACTω, whence it is not provable in FL (it is *-free). By (CUT) and (·-R), $(\gamma(G))^n \cdot \gamma(G) \Rightarrow s$ is not provable in FL The term $(\gamma(G))^n \cdot \gamma(G)$ is equivalent in FL to the disjunction of all terms $\beta_{i_1} \cdot \cdots \cdot \beta_{i_{n+1}}$ such that $(i_1, \ldots, i_{n+1}) \in [k]^{n+1}$. By (∨−L), (∨−R) and (CUT), a sequent $\gamma_1 \vee \ldots \vee \gamma_m \Rightarrow \gamma$ is provable in FL iff all sequents $\gamma_i \Rightarrow \gamma$, for $i = 1, \ldots, m$, are provable in FL. Consequently, there exists a sequence $(i_1, \ldots, i_{n+1}) \in [k]^{n+1}$ such that $\beta_{i_1} \cdot \cdots \cdot \beta_{i_{n+1}} \Rightarrow s$ is not provable in FL. The latter sequent does not contain operation symbols other than $\wedge, \cdot, /, \backslash$, so it is not valid in RACTL, by lemma 2. Consequently, $(\gamma(G))^n, \gamma(G) \Rightarrow s$ is not valid in RACTL. Then, $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is not valid in RACTL (we use the fact that $f(\alpha^n) \subseteq f(\alpha^*)$, for any assigment f and any formula α). Using lemma 1, we obtain: $L(G) = \Sigma^+$ iff $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is valid in RACTL. □

For RACTA, we need a modified reduction. We use a lemma, first proved in [4].

Lemma 3. *Let $\alpha_1, \ldots, \alpha_n$ be types, and let s be a variable. Then, $\alpha_1, \ldots, \alpha_n \Rightarrow s$ is provable in L iff $s/(s/\alpha_1), \ldots, s/(s/\alpha_n) \Rightarrow s$ is provable in L.*

Proof. We outline the proof. A type α/Γ is defined by induction on the length of Γ: $\alpha/\epsilon = \alpha$, $\alpha/(\Gamma\beta) = (\alpha/\beta)/\Gamma$. So, $p/(qr) = (p/r)/q$. We consider the $(/)$−fragment of L. One shows: if $(s/\beta_1 \ldots \beta_k), \Delta \Rightarrow s$ is provable in this system, then there exist $\Delta_1, \ldots, \Delta_k$ such that $\Delta = \Delta_1 \ldots \Delta_k$ and, for each $i = 1, \ldots, k$, $\Delta_i \Rightarrow \alpha_i$ is provable (use induction on cut-free proofs; the converse implication also holds, by (I), (/−L)).

The 'only if' part of the lemma holds, by applying (/−R), (I), (/−L) n times. Now, assume that the right-hand sequent is provable. Denote $\beta_i = s/(s/\alpha_i)$. By the above paragraph, $\beta_2, \ldots, \beta_n \Rightarrow s/\alpha_1$ is provable, so $\beta_2, \ldots, \beta_1, \alpha_1 \Rightarrow s$ is provable (the rule (/−R) is reversible, by (CUT) and the provable sequent $\alpha/\beta, \beta \Rightarrow \alpha$). Repeat this step $n - 1$ times. □

Let $G = (\Sigma, I, s)$ be a Lambek categorial grammar. We construct a Lambek categorial grammars $G' = (\Sigma, I', s)$ such that I' assigns $s/(s/\alpha)$ to $a_i \in \Sigma$ iff G assigns α to a_i. By lemma 3, $L(G') = L(G)$. For G', we construct terms $(\alpha_j^i)'$, $(\beta_i)'$ and $\gamma(G')$ in a way fully analogous to the construction of α_j^i, β_i and $\gamma(G)$. Now, the term $(\beta_i)'$ is of the form:

$$(s/(s/\alpha_1^i)) \wedge \ldots \wedge (s/(s/\alpha_{n_i}^i)) \ . \tag{25}$$

Using the equation $(a/b) \wedge (a/c) = a/(b \vee c)$, valid in residuated lattices, we can transform the above term into an equivalent (in FL) $\wedge-$free term:

$$s/[(s/\alpha_1^i) \vee \ldots \vee (s/\alpha_{n_i}^i)]. \tag{26}$$

Let $\delta(G')$ be the term arising from $\gamma(G')$ by transforming each constituent $(\beta_i)'$ as above. Then, $f(\delta(G')) = f(\gamma(G'))$, for any assignment f.

Theorem 2. *Eq(RACTA) is Π_1^0-hard.*

Proof. $L(G) = \Sigma^+$ iff $L(G') = \Sigma^+$. As in the proofs of lemma 1 and theorem 1, one shows that the second condition is equivalent to: $(\gamma(G'))^*, \gamma(G') \Rightarrow s$ is valid in RACTL. The latter condition is equivalent to: $(\delta(G'))^*, \delta(G') \Rightarrow s$ is valid in RACTL. But the latter sequent is $\wedge-$free, whence it is valid in RACTL iff it is valid in RACTA. □

We can also eliminate $\vee$ (preserving $\wedge$). Using the equation $(a \vee b)^* = (a^*b)^*a^*$, valid in all Kleene algebras, we can transform $(\gamma(G))^*$ into an equivalent (in ACTω) term $\phi(G)$, containing $^*, \wedge, \cdot, /$ only. Then, $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is valid in RACTL iff $\phi(G), \gamma(G) \Rightarrow s$ is valid in RACTL iff $\phi(G) \Rightarrow s/\gamma(G)$ is valid in RACTL, and $s/\gamma(G)$ is equivalent to a $\vee-$free term (see the equation between (25) and (26)). Since $a \leq b$ iff $a \wedge b = a$, then we can reduce $L(G) = \Sigma^+$ to a $\vee-$free equation.

Corollary 1. *The $\vee-$free fragment of Eq(RACTL) is Π_1^0-hard.*

We have found a lower bound for the complexity of Eq(RACTL): it is at least Π_1^0. We did not succeed in determining the upper bound. Both 1 and $\vee$ cause troubles. In section 2, we have shown a sequent with 1 which is valid in RACTL, but not valid in ACTL*. According to the author's knowledge, the precise complexity of the equational theory of relational residuated lattices (upper semilattices) is not known; it must be Σ_1^0, since valid equations can be faithfully interpreted as valid formulas of first-order logic.

We can show some Π_1^0-complete fragments of Eq(RACTL). For instance, the set of all sequents of the form $\alpha, \gamma^*, \beta \Rightarrow p$, with α, β, γ being finite disjunctions of $(/, \backslash, \wedge)-$terms, valid in RACTL is Π_1^0-complete. This sequent is valid iff, for all $n \in \omega$, $\alpha, \gamma^n, \beta \Rightarrow \delta$ is valid, and the latter sequents are valid iff they are provable in FL (see the proof of theorem 1). Consequently, this set of sequents is Π_1^0. It is $\Pi_1^{0)}-$hard, again by the proof of theorem 1. This set can be extended as follows.

A term is said to be *good* if it is formed out of $(\wedge, /, \backslash)-$terms by $\cdot$ and * only. A sequent $\Gamma \Rightarrow \alpha$ is said to be *nice* if it is a $(\wedge, \cdot, ^*, /, \backslash)-$sequent, and any negatively occurring term of the form β^* occurs in this sequent within a good term γ, which appears either as an element of Γ, or in a context δ/γ or $\gamma\backslash\delta$. Using the *-elimination theorem [21], one can prove that the set of nice sequents valid in RACTL is Π^0_1-complete.

4 Algebras of regular languages

A language on Σ is a set $L \subseteq \Sigma^*$. $P(\Sigma^*)$ is the set of all languages on Σ; it is a complete action lattice with operations and designated elements, defined as follows: $L_1 \vee L_2 = L_1 \cup L_2$, $L_1 \wedge L_2 = L_1 \cap L_2$, $L_1 \cdot L_2 = \{xy : x \in L_1, y \in L_2\}$, $1 = \{\epsilon\}$, $0 = \emptyset$, $L^0 = \{\epsilon\}$, $L^{n+1} = L^n \cdot L$, $L^* = \bigcup_{n\in\omega} L^n$, and:

$$L_1/L_2 = \{x \in \Sigma^* : \{x\} \cdot L_2 \subseteq L_1\} \ , \tag{27}$$

$$L_1\backslash L_2 = \{x \in \Sigma^* : L_1 \cdot \{x\} \subseteq L_2\} \ . \tag{28}$$

By LAN we denote the class of all action lattices of the form $P(\Sigma^*)$, for finite alphabets Σ. We add symbols from Σ to the language of ACTω as new individual constants. Regular expressions on Σ can be defined as variable-free terms without meet and residuals. An assignment $L(a) = \{a\}$, for $a \in \Sigma$, is uniquely extended to all regular expressions; it is a homomorphism from the term algebra to $P(\Sigma^*)$. Languages of the form $L(\alpha)$, α is a regular expression on Σ, are called *regular languages* on Σ. By REGL(Σ) we denote the set of all regular languages on Σ. It is well-known that REGL(Σ) is a subalgebra of the action lattice $P(\Sigma^*)$, whence it is a *-continuous action lattice. By REGLAN we denote the class of all action lattices REGL(Σ), for finite alphabets Σ.

We will show that Eq(REGLAN) is Π^0_1-complete. It is quite easy to show that Eq(REGLAN) is Π^0_1. Since regular languages are effectively closed under meet and residuals, $L(\alpha)$ can be computed for all variable-free terms α with individual constants from Σ. An equation $\alpha = \beta$ is valid in REGLAN iff $L(\sigma(\alpha))) = L(\sigma(\beta))$, for all finite alphabets Σ and all substitutions σ assigning regular expressions on Σ to variables.

We note that Eq(RACTL) is different from Eq(REGLAN) and Eq(LAN). The sequent $p, 1/p \Rightarrow 1$ is valid in LAN, and consequently, in REGLAN. Let f be an assignment of terms in $P(\Sigma^*)$. If $f(p) = \emptyset$, then $f(p, 1/p) = \emptyset$. If $f(p) = \{\epsilon\}$, then $f(1/p) = \{\epsilon\}$ and $f(p, 1/p) = \{\epsilon\} = f(1)$. Otherwise $f(1/p) = \emptyset$ and $f(p, 1/p) = \emptyset$. This sequent is not valid in RACTL. Let $U = \{a, b\}$, $a \neq b$, and $f(p) = \{(a, b)\}$. Then, $f(1/p) = \{(a, b), (b, a), (b, b)\}$, so $f(p, 1/p) = \{(a, a), (a, b)\}$ is not contained in I_U.

In [5], it has been shown that the $(\wedge, /, \backslash)-$fragment of FL possesses finite model property. The proof yields, actually, the completeness of this fragment with respect to so-called co-finite models $(P(\Sigma^*), f)$ such that $f(p)$ is a co-finite subset of Σ^*, for any variable p. Then, $f(p)$ is a regular language on Σ. We obtain the following lemma. The proof is a modification of the proof of finite model property of this fragment, given in [7].

Lemma 4. *Let $\Gamma \Rightarrow \alpha$ be a $(\wedge, /, \backslash)-$sequent. Then, $\Gamma \Rightarrow \alpha$ is provable in FL iff it is valid in REGLAN.*

Proof. The 'only if' part is obvious. For the 'if' part, assume that $\Gamma \Rightarrow \alpha$ is not provable. Let T be the set of all subterms appearing in this sequent. We consider languages on the alphabet T. An assignment f_n, $n \in \omega$, is defined as follows: for any variable p, $f_n(p)$ equals the set of all $\Delta \in T^*$ such that either $v(\Delta) > n$, or $\Delta \Rightarrow p$ is provable ($v(\Delta)$ denotes the total number of occurrences of variables in Δ). As usual, f_n is extended to a homomorphism from the term algebra to $P(T^*)$. Since all languages $f_n(p)$ are co-finite, then all languages $f_n(\beta)$ are regular. If $\Delta \in T^*$, $v(\Delta) > n$, then $\Delta \in f_n(\beta)$, for all terms β (easy induction on β).

By induction on $\beta \in T$, we prove: (i) if $v(\Delta) \leq n - v(\beta)$ and $\Delta \in f_n(\beta)$, then $\Delta \Rightarrow \beta$ is provable, (ii) if $v(\beta) \leq v(\Delta)$ and $\Delta \Rightarrow \beta$ is provable, then $\Delta \in f_n(\beta)$. For $\beta = p$, (i) and (ii) follow from the definition of f_n.

Let $\beta = \gamma/\delta$. Assume $v(\Delta) \leq n - v(\beta)$ and $\Delta \in f_n(\beta)$. Since $v(\delta) \leq v(\delta)$, then $\delta \in f_n(\delta)$, by (I) and the induction hypothesis (use (ii)). So, $(\Delta\delta) \in f_n(\gamma)$, by the definition of residuals in $P(T^*)$. Since $v(\Delta\delta) \leq n - v(\gamma)$, then $\Delta, \delta \Rightarrow \gamma$ is provable (use (i)). By $(/-\mathrm{R})$, $\Delta \Rightarrow \beta$ is provable. Assume that $v(\beta) \leq v(\Delta)$ and $\Delta \Rightarrow \beta$ is provable. By the reversibility of $(/-\mathrm{R})$, $\Delta, \delta \Rightarrow \gamma$ is provable. Let $\Phi \in f_n(\delta)$. Case 1: $v(\Phi) > n - v(\delta)$. Then, $v(\Delta\Phi) > n$, whence $(\Delta\Phi) \in f_n(\gamma)$. Case 2: $v(\Phi) \leq n - v(\delta)$. Then, $\Phi \Rightarrow \delta$ is provable, by the induction hypothesis (use (i)), and consequently, $\Delta, \Phi \Rightarrow \gamma$ is provable, by (CUT). Since $v(\gamma) \leq v(\Delta\Phi)$, then $(\Delta\Phi) \in f_n(\gamma)$, by the induction hypothesis (use (ii)). So, $\Delta \in f_n(\beta)$. The case $\beta = \delta\backslash\gamma$ is dual.

Let $\beta = \gamma \wedge \delta$. Assume $v(\Delta) \leq n - v(\beta)$ and $\Delta \in f_n(\beta)$. Then, $v(\Delta) \leq n - v(\gamma)$ and $\Delta \in f_n(\gamma)$. Also $v(\Delta) \leq n - v(\delta)$ and $\Delta \in f_n(\delta)$. By the induction hypothesis, $\Delta \Rightarrow \gamma$ and $\Delta \Rightarrow \delta$ are provable, and consequently, $\Delta \Rightarrow \beta$ is provable, by $(\wedge-\mathrm{R})$. Assume that $v(\beta) \leq v(\Delta)$ and $\Delta \Rightarrow \beta$ is provable. Since $\beta \Rightarrow \gamma$ and $\beta \Rightarrow \delta$ are provable, by (I) and $(\wedge-\mathrm{L})$, then $\Delta \Rightarrow \gamma$ and $\Delta \Rightarrow \delta$ are provable, by (CUT). We have $v(\gamma) \leq v(\Delta)$ and $v(\delta) \leq v(\Delta)$, and consequently, $\Delta \in f_n(\gamma)$ and $\Delta \in f_n(\delta)$, by the induction hypothesis, which yields $\Delta \in f_n(\beta)$.

Take $n = v(\Gamma \Rightarrow \alpha)$. Let $\Gamma = \alpha_1 \ldots \alpha_k$. Since $v(\alpha_i) \leq v(\alpha_i)$, then $\alpha_i \in f_n(\alpha_i)$, by (I) and (ii). Consequently, $\Gamma \in f_n(\Gamma)$. Since $v(\Gamma) = n - v(\alpha)$, then $\Gamma \notin f_n(\alpha)$, by the assumption and (i) (this also holds for $\Gamma = \epsilon$). Consequently, $\Gamma \Rightarrow \alpha$ is not valid in REGLAN. □

Theorem 3. *Eq(REGLAN) is Π^0_1-complete.*

Proof. We know that this set is Π^0_1. We show that it is Π^0_1-hard. We return to lemma 1 in section 3. We show that $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is provable in ACTω iff this sequent is valid in REGLAN. The implication $(\Rightarrow)$ is obvious. To prove $(\Leftarrow)$ assume that $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is not provable in ACTω. As in the proof of theorem 1, we show that there exists a sequence $(i_1, \ldots, i_n) \in [k]^n$, $n \geq 1$, such that $\beta_{i_1} \cdot \cdots \cdot \beta_{i_n} \Rightarrow s$ is not provable in FL. By $(\cdot-\mathrm{L})$, $\beta_{i_1}, \ldots, \beta_{i_n} \Rightarrow s$ is not provable in FL. By lemma 4, the latter sequent is not valid in REGLAN.

As in the proof of theorem 1, we show that $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is not valid in REGLAN. So, $L(G) = \Sigma^+$ iff $(\gamma(G))^*, \gamma(G) \Rightarrow s$ is valid in REGLAN. □

We note that Eq(LAN) belongs to a higher complexity class. The Horn formulas valid in LAN can be expressed by equations valid in LAN. Notice that $\alpha \leq \beta$ is true iff $1 \leq \beta/\alpha$ is true. Also the conjunction of formulas $1 \leq \alpha_i$, $i = 1, \ldots, n$, is true iff $1 \leq \alpha_1 \wedge \cdots \wedge \alpha_n$ is true. Finally, the implication 'if $1 \leq \alpha$ then $1 \leq \beta$' is true iff $1 \wedge \alpha \leq \beta$ is true.

The Horn theory of LAN, restricted to $(/, \backslash)-$terms, is Σ^0_1-complete [3]. The proof of theorem 3 yields the Π^0_1-hardness of Eq(LAN); so, it is not Σ^0_1. If it were Π^0_1, then this restricted Horn theory of LAN would be recursive. So, Eq(LAN) is neither Π^0_1, nor Σ^0_1.

In [17, 11] the Horn theory of KA* and the Horn theory of RKA are shown to be Π^1_1-complete. This yields a lower bound for the complexity of Horn theories of ACTA* and RACTA (every *-continuous Kleene algebra is embeddable into a complete, whence *-continuous, action lattice [9]).

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