Representation of Residuated Semigroups in Some Algebras of Relations (The Method of Canonical Models)

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Abstract

We prove some theorems on representation of residuated semigroups and monoids in algebras of binary relations. One of them has been proved in Andréka and Mikulás [3], and the other are new, though also closely related to some results in [3]. The main novelty of this paper is a simple method of proof, based upon a construction of canonical models for Lambek calculi by means of labeled formulas, whereas [3] uses graph-theoretic constructions. We handle labeled formulas in a way similar to Kurtonina [15] and the second author [14], but we make no use of Labeled Deductive Systems, which is an essential simplification.

1 Introduction and preliminaries

Residuated semigroups and monoids are most general algebraic models for Lambek calculi **L** and **L1**, respectively, stemming from the classical paper [16] of J. Lambek. These calculi are deductive systems underlying an important class of categorial grammars [8, 18]. They are also studied in logic as basic noncommutative substructural or linear logics (see Ono [19], Abrusci [2]). Linguistics is mainly interested in L-models for these calculi which are based on algebras of languages over a finite alphabet [4]. More general structures are powerset frames over arbitrary semigroups (monoids); the strong completeness of **L** and **L1** with respect to the latter models was proved in [6] with applying a special system ND which might be considered as a Labeled

Deductive System (LDS) in the sense of Gabbay [11]. It has been shown in [7, 8] that these methods yield, actually, representation theorems for residuated semigroups and monoids with respect to powerset frames over arbitrary semigroups and monoids, respectively. In [13], similar methods were used to prove an analogous representation theorem for abstract residuated algebras.

Dynamic interpretations of Lambek calculi, stemming from Hoare and Jifeng [12] and being considered by van Benthem [4], suggest models consisting of binary relations on some universe (also see Orłowska [20] for relational interpretations of modal logics). The completeness of **L** and **L1** with respect to relational models was proved in Andréka and Mikulás [3]; the key steps in this proof are representation theorems for residuated semigroups and monoids with respect to algebras of all subrelations of a strict partial ordering and all binary relations on some universe, respectively. These representation theorems are obtained by a direct construction of the desired embedding and the target relational frame by some graph-theoretic techniques. Similar methods were used in Szczerba [24] to prove a representation theorem for residuated groupoids with respect to some restricted relational frames. Other proofs of the weak completeness of **L** with respect to relational models were given by Pankrat'ev [21] (who applied system ND) and Pentus [22] (who proved the weak completeness of **L** with respect to algebras of subrelations of the natural ordering of integers).

In this paper we present a different way for proving representation theorems for residuated semigroups and monoids with respect to appropriate relational frames. We construct canonical models, based upon these frames, for arbitrary axiomatic extensions of **L** and **L1**. These models are constructed as unions of transfinite chains of so-called descriptions (a kind of partial models, introduced in Venema [25] for some labeled versions of the nonassociative Lambek calculus). The construction involves labeled formulas $ab : A$, but it does not use LDSs of Gabbay [11]. In a sense, our results also show that the application of LDSs is not essential in proofs of the 'relational completeness' of **L1** and **L**, given in [15] and [14]. The strong completeness of **L** and **L1** with respect to appropriate relational models is a direct consequence of our results.

Now, we recall basic notions concerning Lambek calculi and their models. Let Pr be a nonempty set of propositional variables, here called *atomic types*. The set $T(Pr)$, of types on Pr, is the smallest set containing Pr and being closed under the following formation rule: if A, B are types, then also $(A \circ B)$, $(A \to B)$ and $(A \leftarrow B)$ are types. *Sequents* are of the form $\Gamma \vdash A$, where Γ is a finite string of types, and A is a type. The system **L1** is based upon the following axioms and inference rules:

$$\text{(Ax)}\ A \vdash A,$$

$$\text{(L}\circ\text{)}\ \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \circ B, \Delta \vdash C}, \qquad \text{(R}\circ\text{)}\ \frac{\Gamma \vdash A;\ \Delta \vdash B}{\Gamma, \Delta \vdash A \circ B},$$

$$\text{(L}\rightarrow\text{)}\ \frac{\Gamma, B, \Delta \vdash C;\ \Phi \vdash A}{\Gamma, \Phi, A \rightarrow B, \Delta \vdash C}, \ \text{(R}\rightarrow\text{)}\ \frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B},$$

$$\text{(L}\leftarrow\text{)}\ \frac{\Gamma, B, \Delta \vdash C;\ \Phi \vdash A}{\Gamma, B \leftarrow A, \Phi, \Delta \vdash C}, \ \text{(R}\leftarrow\text{)}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash B \leftarrow A},$$

$$\text{(CUT)}\ \frac{\Gamma, A, \Delta \vdash B;\ \Phi \vdash A}{\Gamma, \Phi, \Delta \vdash B}.$$

The system **L** is restricted to sequents $\Gamma \vdash A$ with $\Gamma \neq \Lambda$ (Λ denotes the empty string). Thus, $\Gamma \neq \Lambda$ is required in rules (R$\rightarrow$) and (R$\leftarrow$). In both systems, rule (CUT) can be eliminated, but it remains necessary for axiomatic extensions of these systems. **L1** is a nonconservative supersystem of **L**, since, for $p \in Pr$, $(p \rightarrow p) \rightarrow p \vdash p$ is derivable in **L1** but not in **L**. We write $\Gamma \vdash_S A$ for 'A is derivable in system S'.

L is the original Syntactic Calculus of Lambek [16]. **L1** is its natural extension appropriate to languages with the empty string. Also, **L1** is a conservative subsystem of noncommutative linear logics of Yetter [26] and Abrusci [1] (see Abrusci [2]).

A residuated semigroup is a structure $\mathcal{M} = (M, \circ, \rightarrow, \leftarrow, \leq)$, such that $(M, \leq)$ is a nonempty poset, $(M, \circ)$ is a semigroup, and $\rightarrow$, $\leftarrow$ are binary operations on M, satisfying the equivalences:

$$\text{(RES)}\ b \leq a \rightarrow c \text{ iff } a \circ b \leq c \text{ iff } a \leq c \leftarrow b,$$

for all $a, b, c \in M$. One easily shows that $(M, \circ, \leq)$ is, actually, a partially ordered semigroup, that means, $\circ$ is monotone in both arguments. If $(M, \circ)$ has the unit, then $\mathcal{M}$ is called a residuated monoid.

An RS-model is a pair $(\mathcal{M}, \varphi)$ such that $\mathcal{M}$ is a residuated semigroup, and $\varphi : Pr \mapsto M$. One extends the assignment φ to a mapping $\varphi : T(Pr) \mapsto M$ in a standard way. Sequent $A_1 \ldots A_n \vdash A$ is *true* in the model, if:

$$\varphi(A_1) \circ \cdots \circ \varphi(A_n) \leq \varphi(A).$$

RS1-models are defined as RS-models except that $\mathcal{M}$ is a residuated monoid. Sequent $\vdash A$ is *true* in the RS1-model $(\mathcal{M}, \varphi)$, if $1 \leq \varphi(A)$.

If X is a set of sequents, then $\mathbf{L1}(X)$ denotes the system $\mathbf{L1}$ enriched by the sequents from X as new axioms. $\mathbf{L}(X)$ is defined in a similar way (now, we assume sequents from X have nonempty antecedents). It is known that, for any set X, the sequents derivable in $\mathbf{L1}(X)$ are precisely the sequents true in all RS1-models in which all sequents from X are true, and an analogous strong completeness theorem holds for $\mathbf{L}$ and RS-models [6]. Actually, we need merely the soundness, which is easy: all sequents derivable in $\mathbf{L1}(X)$ are true in every RS1-model in which all sequents from X are true, and similarly for $\mathbf{L}(X)$ and RS-models. The strong completeness follows from our lemmas on canonical models.

For $R \subseteq U^2$, we consider the structure $\mathcal{M}(R) = (P(R), \circ, \rightarrow, \leftarrow, \subseteq)$ such that $P(R)$ is the set of all subrelations of R, and operations $\circ, \rightarrow, \leftarrow$ are defined as follows:

$$S \circ T = \{(x,y) \in R : \exists z((x,z) \in S \text{ and } (z,y) \in T)\},$$

$$S \rightarrow T = \{(x,y) \in R : \forall z(\text{ if } (z,x) \in S \text{ then } (z,y) \in T)\},$$

$$T \leftarrow S = \{(x,y) \in R : \forall z(\text{ if } (y,z) \in S \text{ then } (x,z) \in T\},$$

for $S, T \subseteq R$.

Fact 1 *If R is transitive, then $\mathcal{M}(R)$ is a residuated semigroup. If R is reflexive and transitive, then $\mathcal{M}(R)$ is a residuated monoid with the unit I_U (i.e. the identity relation on U).*

PROOF. Exercise.

In this paper we prove the following representation theorems.

Theorem 1 *For every residuated semigroup $\mathcal{M}$, there exists a strict linear ordering relation $\prec$ on some universe U such that $\mathcal{M}$ is isomorphically embeddable into the relational frame $\mathcal{M}(\prec)$.*

Andréka and Mikulás [3] prove a slightly weaker result in which $\prec$ is a strict partial ordering. Theorem 1 cannot be inferred from the latter and the fact that every partial ordering can be extended to a linear ordering, since the extension of R may essentially affect the values of $S \rightarrow T$ and $T \leftarrow S$.

Theorem 2 *For every residuated monoid $\mathcal{M}$, there exists an universe U such that $\mathcal{M}$ is isomorphically embeddable into the relational frame $\mathcal{M}(U^2)$.*

Theorem 2 has been proved in [3]. The next theorem is new.

Theorem 3 *For every residuated monoid $\mathcal{M}$ with unit 1, satisfying:*

(C) for all $a, b \in M$, $1 \leq a \circ b$ entails $1 \leq a$ and $1 \leq b$,

there exists a linear ordering $\preceq$ on some universe U such that $\mathcal{M}$ is isomorphically embeddable into the relational frame $\mathcal{M}(\preceq)$.

In these theorems, the required monomorphism h is supposed to preserve all operations and the ordering, that means:

$$a \leq b \text{ iff } h(a) \subseteq h(b),$$

for all $a, b \in M$. In theorems 2 and 3, we require h satisfy the condition:

$$\text{(C1) } 1 \leq a \text{ iff } I_U \subseteq h(a),$$

for all $a \in M$, but we *do not require* $h(1) = I_U$.

Our proofs of these theorems involve canonical models for arbitrary axiomatic extensions of **L** and **L1**. The existence of such models is stated in the following lemmas.

Lemma 1 *For every set X, of sequents with nonempty antecedents, there exist a strict linear ordering $\prec$ on some universe U and an assignment $\varphi : Pr \mapsto P(\prec)$ such that the sequents derivable in* $\mathbf{L}(X)$ *are precisely the sequents true in model $(\mathcal{M}(\prec), \varphi)$.*

Lemma 2 *For every set X, of sequents, there exist an universe U and an assignment $\varphi : Pr \mapsto P(U^2)$ such that the sequents derivable in* $\mathbf{L1}(X)$ *are precisely the sequents true in model $(\mathcal{M}(\preceq), \varphi)$.*

The third lemma refers to system $\mathbf{L1}^+$ which is **L1** plus the new rules:

$$(\vdash \circ)\ \frac{\vdash A \circ B}{\vdash A},\ \frac{\vdash A \circ B}{\vdash B}.$$

$\mathbf{L1}^+(X)$ is defined as above. Again, it is easy to prove that all sequents derivable in $\mathbf{L1}^+(X)$ are true in every RS1-model in which all sequents from X are true and the unit 1 satisfies (C).

Lemma 3 *For every set X, of sequents, there exist a linear ordering $\preceq$ on some universe U and an assignment $\varphi : Pr \mapsto P(\preceq)$ such that the sequents derivable in* $\mathbf{L1}^+(X)$ *are precisely the sequents true in model $(\mathcal{M}(\preceq), \varphi)$.*

Assuming these lemmas we prove theorems 1, 2 and 3.

Proof of Theorem 1. Let $\mathcal{M} = (M, \circ, \rightarrow, \leftarrow, \leq)$ be a residuated semigroup. To each element $a \in M$ we assign a different atomic type p_a, and we put $Pr = \{p_a : a \in M\}$. An assignment $\mu : Pr \mapsto M$ is defined by setting: $\mu(p_a) = a$, and it is homomorphically extended to a mapping $\mu : T(Pr) \mapsto M$. Let X be the set of all sequents (whose atomic types are in Pr) true in model $(\mathcal{M}, \mu)$.

Let model $(\mathcal{M}(\prec), \varphi)$ be as in lemma 1. We define a mapping $h : M \mapsto P(\prec)$, by setting: $h(a) = \varphi(p_a)$, for $a \in M$. We prove that h is a monomorphism of $\mathcal{M}$ into $\mathcal{M}(\prec)$.

We show that h preserves the operations. First, observe:

$$\mu(p_{a\circ b}) = a \circ b = \mu(p_a) \circ \mu(p_b),$$

$$\mu(p_{a\rightarrow b}) = a \rightarrow b = \mu(p_a) \rightarrow \mu(p_b),$$

$$\mu(p_{b\leftarrow a}) = b \leftarrow a = \mu(p_b) \leftarrow \mu(p_a),$$

for all $a, b \in M$, and consequently, X contains the sequents:

$$p_{a\circ b} \vdash p_a \circ p_b,\ p_a \circ p_b \vdash p_{a\circ b},$$

$$p_{a\rightarrow b} \vdash p_a \rightarrow p_b,\ p_a \rightarrow p_b \vdash p_{a\rightarrow b},$$

$$p_{b\leftarrow a} \vdash p_b \leftarrow p_a,\ p_b \leftarrow p_a \vdash p_{b\leftarrow a},$$

for all $a, b \in M$. Therefore, these sequents are true in model $(\mathcal{M}(\prec), \varphi)$. We obtain:

$$h(a \circ b) = \varphi(p_{a\circ b}) = \varphi(p_a \circ p_b) = \varphi(p_a) \circ \varphi(p_b) = h(a) \circ h(b),$$

and similar calculations yield:

$$h(a \rightarrow b) = h(a) \rightarrow h(b),\ h(b \leftarrow a) = h(b) \leftarrow h(a),$$

for all $a, b \in M$.

We show that h preserves the ordering. First, observe:

$$a \leq b \text{ iff } p_a \vdash_{\mathbf{L}(X)} p_b,$$

for all $a, b \in M$. The 'only if' part holds, by the definition of X. We prove the 'if' part. Assume $p_a \vdash_{\mathbf{L}(X)} p_b$. Since all sequents from X are true in $(\mathcal{M}, \mu)$, then $p_a \vdash p_b$ is true in $(\mathcal{M}, \mu)$, and consequently:

$$a = \mu(p_a) \leq \mu(p_b) = b.$$

Thus, we infer the equivalence:

$$a \leq b \text{ iff } p_a \vdash p_b \text{ is true in } (\mathcal{M}(\prec), \varphi),$$

for all $a, b \in M$, which yields:

$$a \leq b \text{ iff } \varphi(p_a) \subseteq \varphi(p_b) \text{ iff } h(a) \subseteq h(b),$$

for all $a, b \in M$.

Then, h is a monomorphism, since $h(a) = h(b)$ entails $h(a) \subseteq h(b)$ and $h(b) \subseteq h(a)$, which yields $a \leq b$ and $b \leq a$, and this entails $a = b$. $\Box$

PROOF OF THEOREM 2. The above reasoning can be repeated with replacing **L** by **L1**. We prove (C1): $1 \leq a$ iff $1 \leq \mu(p_a)$ iff $\vdash_{\mathbf{L1}(X)} p_a$ iff $\vdash p_a$ is true in $(\mathcal{M}(U^2), \varphi)$ iff $I_U \subseteq \varphi(p_a)$ iff $I_U \subseteq h(a)$. $\Box$

PROOF OF THEOREM 3. The above reasoning can be repeated with replacing **L1** with $\mathbf{L1}^+$. Notice that I_U is the unit of $\mathcal{M}(\preceq)$. $\Box$

Lemma 1 entails the strong completeness of **L** with respect to models based on relational frames $\mathcal{M}(\prec)$ such that $\prec$ is a strict linear ordering, lemma 2 entails the strong completeness of **L1** with respect to models based on relational frames $\mathcal{M}(U^2)$, and lemma 3 entails the strong completeness of $\mathbf{L1}^+$ with respect to models based on relational frames $\mathcal{M}(\preceq)$ such that $\preceq$ is a linear ordering. As corollaries, we obtain the strong completeness of **L**, **L1** and $\mathbf{L1}^+$ with respect to RS-models, RS1-models and RS1-models based on monoids, satisfying (C), respectively. If $\preceq$ is a linear ordering on U, then $\mathcal{M}(\preceq)$ is a residuated monoid, satisfying (C). For $(x, x) \in S \circ T$, $S, T \in P(\preceq)$, entails $(x, z) \in S$ and $(z, x) \in T$, for some $z \in U$; now, $x \preceq z$ and $z \preceq x$, which yields $x = z$, and consequently, $(x, x) \in S$ and $(x, x) \in T$. Thus, $I_U \subseteq S \circ T$ entails $I_U \subseteq S$ and $I_U \subseteq T$, for $S, T \in P(\preceq)$.

Lemmas 1, 2 and 3 will be proved in the next section. The last section indicates further applications of our methods.

2 Labels and descriptions

First, we give a detailed proof of lemma 1. Proofs of lemmas 2 and 3 proceed in a similar way.

We fix Pr and a set X, of sequents (with nonempty antecedents) whose all atomic subtypes are in Pr. We use labeled formulas $ab : A$ such that $A \in T(Pr)$ and a, b are labels.

By *a description* we mean a quadruple $D = (V, \preceq, P, N)$ such that V is a set of labels, $\prec$ is a strict linear ordering on V, and $P, N \subseteq LF(D)$, where

$LF(D)$ denotes the set of all labeled formulas $ab : A$ such that $a, b \in V$, $a \prec b$, and $A \in T(Pr)$. The sets P, N are referred to as the *positive part* and the *negative part*, respectively, of the description.

The description $D = (V, \prec, P, N)$ is said to be *consistent*, if $P \cap N = \emptyset$, and *closed*, if the following condition holds: for every sequent $A_1 \ldots A_n \vdash A$ derivable in $\mathbf{L}(X)$ and all $a_0, \ldots, a_n \in V$, if $a_{i-1}a_i : A_i \in P$, for all $i = 1, \ldots, n$, then $a_0 a_n : A \in P$.

Let D be a description and $P \subseteq LF(D)$. We define the set $C_D(P)$ as the set of all $ab : A \in LF(D)$ such that there exist a sequent $A_1 \ldots A_n \vdash A$ derivable in $\mathbf{L}(X)$ and labels $a_0, \ldots, a_n \in V$, satisfying: $a_0 = a$, $a_n = b$ and $a_{i-1}a_i : A_i \in P$, for all $i = 1, \ldots, n$. Notice that $C_D(P)$ is determined by P, V and $\prec$.

Fact 2 *For any set $P \subseteq LF(D)$, the description D' whose positive part is $C_D(P)$, and the remainder is as in D, is closed.*

Proof. Use (CUT). □

We prove four lemmas concerning extensions of consistent and closed descriptions to new consistent and closed descriptions. For $X = \emptyset$, similar lemmas have been proved in [14], and similar arguments also appear in Kurtonina [15], both heavily using LDSs (also in definitions of basic notions), which are abandoned here.

Lemma 4 *Let $D = (V, \prec, P, N)$ be a consistent and closed description, and let $ab : A \circ B$ be in P. Take a new label $c \notin V$ and define:*
$V' = V \cup \{c\}$,
$\prec' = \prec \cup \{(x, c) : x \preceq a\} \cup \{(c, x) : a \prec x\}$,
$P' = C_{D'}(P \cup \{ac : A,\ cb : B\})$,
$N' = N$.
Then, $D' = (V', \prec', P', N')$ is a consistent and closed description.

Proof. Clearly, $\prec'$ is a strict linear ordering on V', and D' is closed, by fact 2. The definition of P' is correct, although it refers to D', since V' and $\prec'$ have been defined.

We prove that D' is consistent. Suppose not. Then, there exist formulas $a_{i-1}a_i : A_i$, $i = 1, \ldots, n$, belonging to $P \cup \{ac : A,\ cb : B\}$, and type A such that $a_0 a_n : A$ belongs to N and $A_1 \ldots A_n \vdash_{\mathbf{L}(X)} A$. We cannot have $a_{i-1}a_i : A_i \in P$, for all i, since D is consistent. So, there is $i = 1, \ldots, n$

such that $a_{i-1}a_i : A_i$ equals $ac : A$, and $a_ia_{i+1} : A_{i+1}$ equals $cb : B$. We can illustrate the situation be means of a labeled sequent:

$$a_0a_1 : A_1, \dots, ac : A,\ cb : B, \dots, a_{n-1}a_n : A_n \vdash a_0a_n : A,$$

where all antecedent formulas are in $P \cup \{ac : A,\ cb : B\}$, and the consequent formula is in N. Consequently, $A_1 \dots A_{i-1}, A, B, A_{i+2} \dots A_n \vdash_{\mathbf{L}(X)} A$, hence also $A_1 \dots A_{i-1}, A \circ B, A_{i+2} \dots A_n \vdash_{\mathbf{L}(X)} A$, by rule (L$\circ$), and we get a new labeled sequent:

$$a_0a_1 : A_1, \dots, ab : A \circ B, \dots, a_{n-1}a_n : A_n \vdash a_0a_n : A.$$

In this way we can eliminate all occurrences of formulas $ac : A$ and $cb : B$ from the initial labeled sequent (here, this pair cannot occur twice, since labels are linearly ordered, but more occurrences of this pair will be possible in analogous arguments later on). Now, $ab : A \circ B \in P$, and D is closed, hence D is not consistent. Contradiction. $\Box$

Lemma 5 *Let $D = (V, \prec, P, N)$ be a consistent and closed description, and let $ab : A \to B$ be in N. Take a new label $c \notin V$ and define:*
$V' = V \cup \{c\}$,
$\prec' = \prec \cup \{(c, x) : x \in V\}$,
$P' = C_{D'}(P \cup \{ca : A\})$,
$N' = N \cup \{cb : B\}$.
Then, $D' = (V', \prec', P', N')$ is a consistent and closed description.

Proof. The argument is similar. D' could be not consistent, only if there existed a labeled sequent:

$$ca_1 : A,\ a_1a_2 : A_2, \dots, a_{n-1}a_n : A_n \vdash ca_n : B,$$

such that $a_1 = a$, $a_n = b$, $a_{i-1}a_i : A_i \in P$, for all $i = 2, \dots, n$, and $A, A_1 \dots A_n \vdash_{\mathbf{L}(X)} B$. Using rule (R$\to$), we obtain $A_2 \dots A_n \vdash A \to B$ (note that $n \geq 2$, since $a \prec b$, and consequently, $a \neq b$). Thus, D is inconsistent. $\Box$

Lemma 6 *Let $D = (V, \prec, P, N)$ be a consistent and closed description, and let $ba : B \leftarrow A$ be in N. Take a new label $c \notin V$ and define:*
$V' = V \cup \{c\}$,
$\prec' = \prec \cup \{(x, c) : x \in V\}$,
$P' = P \cup \{ac : A\}$,
$N' = N \cup \{bc : B\}$.
Then, $D' = (V', \prec', P', N')$ is a consistent and closed description.

PROOF. The proof is dual to the preceding one. □

Lemma 7 *Let $D = (V, \prec, P, N)$ be a consistent and closed description, and let $ab : A \in LF(D)$, $ab : A \notin P$. Define $V' = V$, $\prec' = \prec$, $P' = P$, $N' = N \cup \{ab : A\}$. Then, $D' = (V', \prec', P', N')$ is a consistent and closed description.*

PROOF. Straightforward. □

We define a description D_X. For any sequent $A_1 \ldots A_n \vdash A$, $n \geq 1$, not derivable in $\mathbf{L}(X)$, we choose different labels $a_0, \ldots, a_n$ (different sequents are assigned different labels), order them in the natural way: $a_0 \prec a_1 \prec \ldots \prec a_n$, and set:
$V =$ the set of all labels produced in this way,
$\prec' =$ a strict linear ordering on V which extends all natural orderings of labels,
$P =$ the set of all formulas $a_{i-1}a_i : A_i$ stemming from particular sequents,
$N =$ the set of all formulas $a_0 a_n : A$ stemming from particular sequents,
$D_X = (V, \preceq, C_{D_X}(P), N)$.

Lemma 8 *D_X is a consistent and closed description.*

PROOF. Only the consistency of D_X requires a proof. Suppose that there is a labeled sequent:

$$a_0 a_1 : A_1, \ldots, a_{n-1} a_n : A_n \vdash a_0 a_n : A$$

such that $a_{i-1}a_i : A_i \in P$, for all $i = 1, \ldots, n$, $a_0 a_n : A \in N$, and $A_1 \ldots A_n \vdash_{\mathbf{L}(X)} A$. Since labels assigned to different sequents are different, then one easily sees that $a_0, \ldots, a_n$ must be assigned to sequent $A_1 \ldots A_n \vdash A$, and consequently, this sequent is not derivable in $\mathbf{L}(X)$. Contradiction. □

To prove lemma 1 we need the following sequents derivable in $\mathbf{L}$:

(1) $A, B \vdash A \circ B$ (use (Ax) and (R$\circ$)),

(2) $A, A \rightarrow B \vdash B$ (use (Ax) and (L$\rightarrow$)),

(3) $B \leftarrow A, A \vdash B$ (use (Ax) and (L$\leftarrow$)).

A description $D = (V, \prec, P, N)$ is said to be *complete*, if $P \cup N = LF(D)$, and *witnessed*, if, for all $a, b \in V$, the following conditions hold:

(W1) if $ab : A \circ B \in P$, then $ac : A \in P$ and $cb : B \in P$, for some $c \in V$, (W2) if $ab : A \rightarrow B \in N$, then $ca : A \in P$ and $cb : B \in N$, for some $c \in V$,

(W3) if $ba : B \leftarrow A \in N$, then $ac : A \in P$ and $bc : B \in N$, for some $c \in V$.

If $D = (V, \prec, P, N)$ and $D' = (V', \prec', P', N')$ are descriptions, then we write $D \sqsubseteq D'$, if $V \subseteq V'$, $\prec \subseteq \prec'$, $P \subseteq P'$ and $N \subseteq N'$.

Lemma 9 *For every consistent and closed description D, there exists a consistent, closed, complete and witnessed description D' such that $D \sqsubseteq D'$.*

PROOF. We use a routine transfinite recursion. Let $D = (V, \prec, P, N)$ be a consistent and closed description. Choose an infinite cardinal $\kappa \geq \mathrm{card}(T(Pr) \cup V)$ and a set W, of new labels, such that $\mathrm{card}(W) = \kappa$. We fix a transfinite sequence $(w_\xi)_{\xi<\kappa}$, of all labels from W (without repetitions). By $\mathcal{F}$ we denote the set of all formulas $ab : A$ such that $A \in T(Pr)$ and $a, b \in V \cup W$. Clearly, $\mathrm{card}(\mathcal{F}) = \kappa$, and we fix a transfinite sequence $(\gamma_\xi)_{\xi<\kappa}$, of all elements of $\mathcal{F}$, such that:

(CF) for all $\gamma \in \mathcal{F}$ and $\xi < \kappa$, there exists $\kappa > \zeta \geq \xi$ such that $\gamma = \gamma_\zeta$.

We define a transfinite chain of descriptions $(D_\xi)_{\xi<\kappa}$ with $D_\xi \sqsubseteq D_{\xi+1}$. We write $D_\xi = (V_\xi, \prec_\xi, P_\xi, N_\xi)$. The definition guarantees $V_\xi \subseteq V \cup W$ and $w_\xi \notin V_\xi$, for all $\xi < \kappa$.

Put $D_0 = D$. For limit ordinals $\lambda < \kappa$, define $V_\lambda = \bigcup_{\xi<\lambda} V_\xi$, and similarly for $\prec_\lambda$, P_λ and N_λ. We define $D_{\xi+1}$ on the basis of D_ξ. We take the labeled formula γ_ξ and consider several cases.

(I) $\gamma_\xi = ab : A \circ B$ is in P_ξ, and there is no $c \in V_\xi$ such that $ac : A \in P_\xi$ and $cb : B \in P_\xi$. We put $c = w_\xi$ and define $D_{\xi+1}$ as in lemma 4.

(II) $\gamma_\xi = ab : A \rightarrow B$ is in N_ξ, and there is no label $c \in V_\xi$ such that $ca : A \in P_\xi$ and $cb : B \in N_\xi$. We put $c = w_\xi$ and define $D_{\xi+1}$ as in lemma 5.

(III) $\gamma_\xi = ba : B \leftarrow A$ is in N_ξ, and there is no label $c \in V_\xi$ such that $ac : A \in P_\xi$ and $bc : B \in N_\xi$. We put $c = w_\xi$ and define $D_{\xi+1}$ as in lemma 6.

(IV) $\gamma_\xi \notin P_\xi \cup N_\xi$, $\gamma_\xi \in LF(D_\xi)$. We define $D_{\xi+1}$ as in lemma 7.

(V) In the remaining cases, we set $D_{\xi+1} = D_\xi$.

Using lemmas 4-7, one easily shows that all descriptions D_ξ are consistent and closed. We define $D' = (V', \prec', P', N')$, where $V' = \bigcup_{\xi<\kappa} V_\xi$, and similarly for $\prec'$, P' and N'. Clearly, D' is consistent and closed. By the construction and (CF), D' is also witnessed and complete. We show the completeness. Assume $\gamma \in LF(D')$, $\gamma \notin P'$. Then, $\gamma = ab : A$ with $a \prec' b$, hence $\gamma \in LF(D_\xi)$, for some $\xi < \kappa$. By (CF), $\gamma = \gamma_\zeta$, for some $\zeta \geq \xi$, and $\gamma \in LF(D_\zeta)$. By (IV), $\gamma \in N_{\zeta+1}$, hence $\gamma \in N'$. $\square$

Lemma 10 *Let $D = (V, \prec, P, N)$ be a consistent, closed, complete and witnessed description. Let an assignment $\varphi : Pr \mapsto P(\prec)$ be defined as follows:*

(M1) $(a, b) \in \varphi(p)$ iff $ab : p \in P$,

for all $a, b \in V$ and $p \in Pr$. Then, for all $a, b \in V$ and $A \in T(Pr)$:

(M2) $(a, b) \in \varphi(A)$ iff $ab : A \in P$.

PROOF. Induction on the complexity of A. (M2) for atomic types p holds, by (M1).

Case $A \circ B$. If $(a, b) \in \varphi(A \circ B)$, then there is $c \in V$ such that $(a, c) \in \varphi(A)$ and $(c, b) \in \varphi(B)$, which yields $ac : A \in P$ and $cb : B \in P$; since $A, B \vdash_{\mathbf{L}} A \circ B$, and D is closed, we obtain $ab : A \circ B \in P$. Assume $ab : A \circ B \in P$. Since D is witnessed, there is $c \in V$ such that $ac : A \in P$ and $cb : B \in P$, which yields $(a, c) \in \varphi(A)$ and $(b, c) \in \varphi(B)$, and consequently, $(a, b) \in \varphi(A \circ B)$.

Case $A \to B$. Assume $ab : A \to B \in P$. Take $c \in V$ such that $(c, a) \in \varphi(A)$, hence $ca : A \in P$. Since $A, A \to B \vdash_{\mathbf{L}} B$, and D is closed, we obtain $cb : B \in P$, and consequently, $(c, b) \in \varphi(B)$. Thus, $(a, b) \in \varphi(A \to B)$. Assume $ab : A \to B \notin P$. If $a \not\prec b$, then $(a, b) \notin \varphi(A \to B)$. So, assume $a \prec b$. Then, $ab : A \to B \in LF(D)$, and consequently, $ab : A \to B \in N$, since D is complete. Since D is witnessed, there exists $c \in V$ such that $ca : A \in P$ and $cb : B \in N$, which yields $(c, a) \in \varphi(A)$ and $(c, b) \notin \varphi(B)$ (we have $cb : B \notin P$, since D is consistent). Therefore, $(a, b) \notin \varphi(A \to B)$.

Case $B \leftarrow A$ is dual. $\Box$

PROOF OF LEMMA 1. Take the description D_X and find a consistent, closed, complete and witnessed description D such that $D_X \sqsubseteq D$. Let $D = (V, \prec, P, N)$. Define φ as above. We prove that the sequents derivable in $\mathbf{L}(X)$ are precisely the sequents true in model $(\mathcal{M}(\prec), \varphi)$.

Let $A_1 \dots A_n \vdash_{\mathbf{L}(X)} A$. Let $(a_{i-1}, a_i) \in \varphi(A_i)$, $i = 1, \dots, n$. By (M2), $a_{i-1}a_i : A_i \in P$, $i = 1, \dots, n$, and consequently, $a_0 a_n : A \in P$, since D is closed, which yields $(a_0, a_n) \in \varphi(A)$. We have shown that $A_1 \dots A_n \vdash A$ is true in the model.

Assume $A_1 \dots A_n \vdash A$ be not derivable in $\mathbf{L}(X)$. By the construction of D_X, there are labels $a_0, \dots, a_n \in V$ such that $a_{i-1}a_i : A_i \in P$, for all $i = 1, \dots, n$, and $a_0 a_n \in N$. By (M2) and the consistency of D, $(a_{i-1}, a_i) \in \varphi(A_i)$, for all $i = 1, \dots, n$, and $(a_0, a_n) \notin \varphi(A)$, hence $A_1 \dots A_n \vdash A$ is not true in the model. $\Box$

The proof of lemma 2 goes along the same lines, but some notions must be changed. *A description* is defined as a triple $D = (V, P, N)$ such that

V is a set of labels, and $P, N \subseteq LF(D)$, where $LF(D)$ denotes the set of all formulas $ab : A$, for $A \in T(Pr)$ and $a, b \in V$. Again, we fix a set Pr and a set X, of sequents whose atomic subtypes are in Pr. The definition of consistency is unchanged, while in the definitions of closedness and C_D one replaces $\mathbf{L}(X)$ by $\mathbf{L1}(X)$. In particular, if $D = (V, P, N)$ is closed, then $aa : A \in P$, for all $a \in V$ and $A \in T(Pr)$ such that $\vdash_{\mathbf{L1}(X)} A$, and similarly, $C_D(P')$ always contains all formulas of that kind.

The analogue of fact 2 obviously holds. The analogues of lemmas 4, 5 and 6 and 7 are obtained by dropping orderings, and proofs are almost the same (now, $a = b$ is possible, for $ab : A \in LF(D)$, but it causes no trouble). The construction of D_X has the extra-clause: for any $A \in T(Pr)$ such that $\vdash A$ is not derivable in $\mathbf{L1}(X)$, we choose a special label a and put $aa : A \in N$. Then, the analogue of lemma 8 holds with the same proof.

Complete and witnessed descriptions are defined as above (we drop orderings), and the analogues of lemmas 9 and 10 admit similar proofs. In the final proof of lemma 2, we must regard sequents $\vdash A$. By the soundness of $\mathbf{L1}(X)$ with respect to RS1-models, if $\vdash_{\mathbf{L1}(X)} A$ then A is true in $(\mathcal{M}(V^2), \varphi)$. Assume $\vdash A$ be not derivable in $\mathbf{L1}(X)$. Then, by the construction of D_X, there is a label $a \in V$ such that $aa : A \in N$, hence $aa : A \notin P$, and we obtain $(a, a) \notin \varphi(A)$, by (M2). Consequently, $I_V \not\subseteq \varphi(A)$, which means that $\vdash A$ is not true in the model.

The proof of lemma 3 mixes some elements of proofs of lemmas 1 and 2 and needs something more. *Descriptions* are quadruples $D = (V, \preceq, P, N)$ such that V is a set of labels, $\preceq$ is a linear ordering on V, and $P, N \subseteq LF(D)$, where $LF(D)$ is the set of all formulas $ab : A$ such that $a, b \in V$, $a \preceq b$ and $A \in T(Pr)$. Consistent, complete and witnessed descriptions are defined as above. In the definition of closedness and C_D, we replace $\mathbf{L}(X)$ by $\mathbf{L1}^+(X)$. Again, if D is closed, and $\vdash A$ is derivable in $\mathbf{L1}^+(X)$, then $aa : A \in P$, for all $a \in V$, and $C_D(P')$ must contain all formulas of that kind.

A description $D = (V, \preceq, P, N)$ is said to be *good*, if, for all $a \in V$ and $A \in T(Pr)$:

$$\text{(G) } aa : A \in P \text{ iff } \vdash_{\mathbf{L1}^+(X)} A.$$

Fact 3 *Let $D = (V, \preceq, P', N)$ be a description, satisfying the 'only if' part of (G). Then, the description $D' = (V, \preceq, C_D(P'), N)$ is closed and good.*

PROOF. One easily shows that D' is closed, and consequently, it satisfies the 'if' part of (G). We prove the 'only if' part. Let $aa : A \in C_D(P')$. Suppose $\vdash A$ be not derivable in $\mathbf{L1}^+(X)$. Then, there exist formulas $a_{i-1}a_i : A_i \in P'$,

$i = 1, \ldots, n$, $n \geq 1$, such that $A_1 \ldots A_n \vdash A$ is derivable in $\mathbf{L1}^+(X)$, $a_0 = a = a_n$. Since labels are linearly ordered, we have $a_i = a$, for all $i = 1, \ldots, n$. By the assumption on D, $\vdash A_i$ is derivable in $\mathbf{L1}^+(X)$, for all $i = 1, \ldots, n$, and consequently, $\vdash A$ is also derivable, by (CUT). Contradiction. $\square$

The analogue of fact 2 can be proved in a stronger form: for every good description $D = (V, \preceq, P, N)$, the description $D' = (V, \preceq, C_D(P), N)$ is good and closed.

The analogue of lemma 4 assumes $a \neq b$, replaces $\prec$ by $\preceq$ and 'description' by 'good description', and defines:

$$\preceq' = \preceq \cup \{(x, c) : x \preceq a\} \cup \{(c, x) : a \preceq x,\ a \neq x\} \cup \{(c, c)\}.$$

The analogues of lemmas 5 and 6 only replace $\prec$ by $\preceq$ and 'description' by 'good description', and add (c, c) in the definition of $\preceq'$. In the analogue of lemma 7 we replace $\prec$ by $\preceq$ and 'description' by 'good description'. Proofs of all these lemmas are as above.

The construction of D_X is the same as in the proof of lemma 1 except that now the outcome ordering $\preceq$ on V is a linear ordering on V whose strict part extends all natural orderings of labels stemming from particular sequents; as in the proof of lemma 2, one adds special labels a, for any sequent $\vdash A$, not derivable in $\mathbf{L1}^+(X)$, and puts $aa : A \in N$. The analogue of lemma 8 claims that D_X is good, consistent and closed. The goodness follows from fact 3, since P trivially satisfies the 'only if' part of (G) (no formula $aa : A$ belongs to P).

The analogue of lemma 9 replaces $\prec$ with $\preceq$ and 'description' by 'good description'. The proof goes along the same lines; all descriptions D_ξ are good, consistent and closed. The only noteworthy difference concerns case (I) of the definition of $D_{\xi+1}$. We must regard the case: $\gamma_\xi = aa : A \circ B$ is in P_ξ. We cannot apply the analogue of lemma 4 (now $a = b$). Since P_ξ is good, $\vdash A \circ B$ is derivable in $\mathbf{L1}^+(X)$. Using rules $(\vdash \circ)$, one derives $\vdash A$ and $\vdash B$ in $\mathbf{L1}^+(X)$, which yields $aa : A \in P_\xi$ and $aa : B \in P_\xi$, again by goodness (or closedness). So, $D_{\xi+1} = D_\xi$.

The analogue of lemma 10 replaces $\prec$ by $\preceq$ and admits the same proof. The final proof of lemma 3 does not essentially differ from that of lemma 2.

3 Final comments

Proofs of representation theorems, presented above, use model-theoretic tools: the key step is a construction of a canonical model for any axiomatic extension of the appropriate variant of the Lambek calculus. This method is essentially different from the method of labeled graphs, applied by Andréka and Mikulás [3], and from the method of cones (filters), applied in Dunn [10] to prove representation theorems for residuated semigroups with respect to so-called concrete algebras (the latter method is an adaptation of the standard technique of representing Boolean and Heyting algebras in some algebras of sets, presented in Rasiowa [23]; see the discussion in [7]).

After one has compared proofs of lemmas 1-3 with proofs of theorems 1-3, then she sees that types involved in the former represent, actually, elements of the initial residuated semigroup (monoid) $\mathcal{M}$, and consequently, labeled formulas $ab : A$ can be related to labeled edges in the proofs of Andréka and Mikulás [3]. Thus, there seem to be deeper connections between these two strategies of proof. Our proofs, however, seem to be simpler, since they abandon sophisticated checkings of 'triangle conditions' for labeling functions; instead, we have lemmas 4-7 on extensions of descriptions, and their proofs refer to simple deductive properties of the Lambek calculus.

Our method of canonical models can be applied to obtain other representation and completeness theorems for related systems and algebraic structures. As in [3], one can consider the Lambek calculus with conjunction $\wedge$, governed by the rules:

$$(\mathrm{L}\wedge)\ \frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C},\ \frac{\Gamma, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C},$$

$$(\mathrm{R}\wedge)\ \frac{\Gamma \vdash A;\ \Gamma \vdash B}{\Gamma \vdash A \wedge B}.$$

This expansion leads to systems $\mathbf{L}\wedge$, $\mathbf{L1}\wedge$ and $\mathbf{L1}^{+}\wedge$.

Algebraic frames for these systems are *lower semilattice residuated semigroups* and *monoids* in which the operation $\wedge$ satisfies the conditions: $a \wedge b \leq a$, $a \wedge b \leq b$, and $c \leq a \wedge b$ if $c \leq a$ and $c \leq b$. In relational frames, one interprets $\wedge$ as the intersection of binary relations.

One can prove that every lower semilattice residuated semigroup is isomorphically embeddable into a relational frame $\mathcal{M}(\prec)$ such that $\prec$ is a strict linear ordering on some universe U. The proof is similar to that of theorem 1; now, one must guarantee that all descriptions appearing in lemmas 4-10 satisfy the condition: if $ab : A$ and $ab : B$ are in P, then $ab : A \wedge B$ is in P. In a similar way, one can prove analogues of theorems 2 and 3 (and, obviously, lemmas 2 and 3, which yields the strong completeness of appropriate variants of the Lambek calculus with respect to relational frames with intersection). We leave details to forthcoming papers.

Another natural area of application of our method is nonassociative variants of the Lambek calculus [17] and the corresponding algebraic structures: residuated groupoids and residuated groupoids with unit. Szczerba [24] introduces restricted relational frames $(P(R), \circ, \rightarrow, \leftarrow, \subseteq)$ such that $R \subseteq U^2$, $\circ$ is defined as for relational frames (see section 1), and $\rightarrow$, $\leftarrow$ are defined as follows:

$$S \rightarrow T = \{(x, y) \in R : \forall z(\text{ if } (z, x) \in S \text{ and } (z, y) \in R \text{ then } (z, y) \in T)\},$$

$$T \leftarrow S = \{(x, y) \in R : \forall z(\text{ if } (y, z) \in S \text{ and } (x, z) \in R \text{ then } (x, z) \in T)\},$$

and proves that every residuated groupoid is isomorphically embeddable into some restricted relational frame, using graph-theoretic methods of [3]. (Restricted relational frames are residuated groupoids, i.e. groupoids satisfying (RES); if R is transitive, they coincide with relational frames $\mathcal{M}(R)$.) The method of canonical models provides an alternative way to prove more representation theorems of that kind.

We leave it as an open problem, if the method of canonical models can be applied to richer systems, as e.g. the full Lambek calculus of Ono [19] and corresponding algebraic structures. It would also be interesting to find relational frames adequate to commutative variants of the Lambek calculus (the logic BCI, fragments of linear logic). Finally, we do not know, if a refinement of the above constructions can yield the completeness of the Lambek calculus with respect to finite relational frames, and precisely, algebras of relations over a finite universe.

References

[1] V.M. Abrusci, Phase semantics and sequent calculus for pure noncommutative classical linear propositional logic, *The Journal of Symbolic Logic* 56 (1991), 1403-1451.

[2] V.M. Abrusci, Lambek Calculus, Cyclic Mutiplicative-Additive Linear Logic, Noncommutative Multiplicative-Additive Linear Logic: language and sequent calculus, in: V.M. Abrusci and C. Casadio (eds.), *Proofs and Linguistic Categories. Applications of Logic to the Analysis and Implementation of Natural Language*, University of Bologna Press, 1996, 21-48.

[3] H. Andréka and S. Mikulás, Lambek Calculus and Its Relational Semantics. Completeness and Incompleteness, *Journal of Logic, Language and Information* 3(1), (1994), 1-37.

[4] J. van Benthem, *Language in Action. Categories, Lambdas and Dynamic Logic*, North-Holland, Amsterdam, 1991.

[5] J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*, Elsevier, Amsterdam, The MIT Press, Cambridge (Mass.), 1997.

[6] W. Buszkowski, Completeness results for Lambek Syntactic Calculus, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 32 (1986), 13-28.

[7] W. Buszkowski, Representation Theorems for Implication Structures, *Bulletin of The Section of Logic* 25, (1996), 152-158.

[8] W. Buszkowski, Mathematical linguistics and proof theory, in [5], 683-736.

[9] K. Došen and P. Schroeder-Heister (eds.), *Substructural Logics*, Claredon Press, Oxford, 1993.

[10] J.M. Dunn, Partial gaggles applied to logics with restricted structural rules, in [9], 63-108.

[11] D. Gabbay, *Labeled Deductive Systems. I*, CIS München, Munich, 1991.

[12] C.A.R. Hoare and H. Jifeng, The weakest prespecification, *Fundamenta Informaticae* IX, (1986), Part I 51-84, Part II 217-262.

[13] M. Kołowska-Gawiejnowicz, Powerset Residuated Algebras and Generalized Lambek Calculus, *Mathematical Logic Quarterly* 43, (1997), 60-72.

[14] M. Kołowska-Gawiejnowicz, A Labelled Deductive System for Relational Semantics of the Lambek Calculus, *Report* 063/1997, Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, 1997.

[15] N. Kurtonina, *Frames and labels. A modal analysis of categorial inference*, Ph.D. Thesis, University of Utrecht, 1995.

[16] J. Lambek, The mathematics of sentence structure, *The American Mathematical Monthly* 65, (1958), 154-170.

[17] J. Lambek, On the calculus of syntactic types, in: R. Jakobson (ed.), *Structure of Language and Its Mathematical Aspects*, AMS, Providence, 1961.

[18] M. Moortgat, Categorial type logics, in [5], 93-177.

[19] H. Ono, Semantics for Substructural Logics, in [9], 259-291.

[20] E. Orłowska, Relational interpretation of modal logics, in: H. Andréka, D. Monk and I. Nemeti (eds.), *Algebraic Logic*, Coll. Math. Soc. J. Bolyai 54, North-Holland, Amsterdam, 1988.

[21] N. Pankrat'ev, Completeness of the Lambek Calculus with respect to relativized relational semantics, *Journal of Logic, Language and Information* 3, (1994), 233-246.

[22] M. Pentus, Lambek Calculus is L-complete, *ILLC Prepublication Series* LP-93-14, University of Amsterdam, 1993.

[23] H. Rasiowa, *An algebraic approach to non-classical logics*, North- Holland, Amsterdam, Polish Scientific Publishers, Warszawa, 1974.

[24] M. Szczerba, Representation theorems for residuated groupoids, in: C. Retoré et al. (eds.), *Proc. of LACL*, Nancy, 1996, to appear.

[25] Y. Venema, Tree models and (labelled) categorial grammar, *Journal of Logic, Language and Information* 5, (1996), 253-277.

[26] D.N. Yetter, Quantales and (Noncommutative) Linear Logic, *The Journal of Symbolic Logic* 55, (1990), 41-64.