Multi-Sorted Residuation

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Abstract. Nonassociative Lambek Calculus (**NL**) is a pure logic of residuation, involving one binary operation (product) and its two residual operations defined on a poset [26]. Generalized Lambek Calculus **GL** involves a finite number of basic operations (with an arbitrary number of arguments) and their residual operations [7]. In this paper we study a further generalization of **GL** which admits operations whose arguments and values can be of different sorts. This logic is called *Multi-Sorted Lambek Calculus* **mL**. We also consider its variants with lattice and boolean operations. We discuss some basic properties of these logics (completeness, decidability, complexity and others) and the corresponding algebras.

1 Introduction

Nonassociative Lambek Calculus (**NL**) was introduced in Lambek [26] as a weaker variant of Syntactic Calculus [25], the latter nowadays called (Associative) Lambek Calculus (**L**). Lambek's motivation for **NL** was linguistic: to block some overgeneration, appearing when sentences are parsed by means of **L**. For example, *John likes poor Jane* and *John likes him* justify the following typing:

$$\text{John, Jane: } n, \text{ likes: } (n\backslash s)/n, \text{ poor: } n/n, \text{ him: } (s/n)\backslash s\,,$$

which yields type s of **John likes poor him* in **L**, but not **NL**.

Besides linguistic interpretations, usually related to *type grammars*, these calculi became popular in some groups of logicians, as basic *substructural logics*. **L** admitting Exchange and sequents $\Rightarrow A$ (i.e. sequents with the empty antecedent) is equivalent to the $\{\otimes, \to\}-$fragment of Linear Logic of Girard, and without Exchange to an analogous fragment of Noncommutative Linear Logic of Abrusci. Full Lambek Calculus (**FL**), i.e. **L** with 1, 0 (optionally) and lattice connectives $\sqcup, \sqcap$, and its nonassociative version **FNL** are treated as basic substructural logics in the representative monograph [11] (**FNL** is denoted **GL** from 'groupoid logic', but we use the latter symbol in a different meaning). Recall that substructural logics are nonclassical logics whose Gentzen-style sequent systems omit some structural rules (Exchange, Weakening, Contraction). This class contains (among others) relevant logics (omit Weakening) and multi-valued logics (omit Contraction); they can be presented as axiomatic extensions of **FL**.

Studies in substructural logics typically focus on associative systems in which *product* $\otimes$ is associative. Nonassociative systems are less popular among logicians, although they are occasionally considered as a close companion of the former. In the linguistic community, some work has been done in Nonassociative Lambek Calculus, treated as a natural framework for parsing structured expressions. This approach is dominating in Moortgat's studies on type grammars; besides nonassociative product and its residuals $\backslash, /$, Moortgat considers different unary modalities and their residuals which allow a controlled usage of certain structural rules [30]. Recently, Moortgat [31] also admits a dual residuation triple, which leads to some Grishin-style nonassociative systems. Nonassociative Lambek Calculus was shown context-free in [5] (the product-free fragment) and [20] (the full system). A different proof was given by Jäger [15], and its refinement yields the polynomial time complexity and the context-freeness of **NL** augmented with (finitely many) assumptions [6].

A straightforward generalization of **NL** admits an arbitrary number of generalized product operations of different arities together with their residuals. The resulting system, called *Generalized Lambek Calculus*, was studied in the author's book (*Logical Foundations of Ajdukiewicz-Lambek Categorial Grammars*, in Polish, 1989) and later papers [6, 9, 7] (also with lattice and boolean operations). In this setting the associative law is not assumed, as not meaningful for non-binary operations.

The present paper introduces a further generalization of this framework: different product operations are not required to act on the same universe. For instance, one may consider an operation $f : A \times B \mapsto C$ with residuals $f^{r,1} : C \times B \mapsto A$ and $f^{r,2} : A \times C \mapsto B$ and another operation $g : A' \times B' \mapsto C'$ with residuals $g^{r,1}, g^{r,2}$. Here A, B, C represent certain ordered algebras: posets, semi-lattices, lattices, boolean algebras etc., and one assumes the residuation law: $f(x, y) \leq_C z$ iff $x \leq_A f^{r,1}(z, y)$ iff $y \leq_B f^{r,2}(x, z)$.

This approach seems quite natural: in mathematics one often meets residuated operations acting between different universes, and such operations can also be used in linguistics (see section 2). The resulting multi-sorted residuation logic extends **NL**, and we show here that it inherits many essential proof-theoretic, model-theoretic and computational properties of **NL**. For instance, without lattice operations it determines a polynomial consequence relation; with distributive lattice or boolean operations the consequence relation remains decidable in opposition to the case of **L**.

The multi-sorted framework can further be generalized by considering categorical notions, but this generalization is not the same as cartesian-closed categories, studied by Lambek and others; see e.g. [28, 27]. Instead of a single category with object-constructors $A \times B, A^B, {}^B A$, corresponding to the algebraic $a \otimes b, a\backslash b, b/a$, one should consider a multicategory whose morphisms are residuated maps. We do not develop this approach here.

In section 2 we define basic notions, concerning residuated maps, and provide several illustrations. In particular, we show how multi-sorted residuated maps can be used in modal logics and linguistics.

In section 3 we consider multi-sorted (heterogeneous) residuation algebras: abstract algebraic models of multi-sorted residuation logics. We discuss canonical embeddings of such algebras into complex algebras of multi-sorted relational frames, which yield some completeness theorems for multi-sorted residuation logics. The multi-sorted perspective enables one to find more uniform proofs of embedding theorems even for the one-sort case.

The multi-sorted residuation logics are defined in section 4; the basic system is *Multi-Sorted Lambek Calculus* **mL**, but we also consider some extensions of it. In general, basic properties of one-sort residuation logics are preserved by multi-sorted logics. Therefore we omit most proofs. Some events, however, only appear in the multi-sorted world (e.g. classical paraconsistent theories).

Some ideas of this paper have been presented in the author's talk 'Many-sorted gaggles' at the conference *Algebra and Coalgebra Meet Proof Theory*, Prague, 2012 [8].

2 Residuated maps

Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be posets. A map $f : P_1 \mapsto P_2$ is said to be *residuated*, if the co-image $f^{-1}[x^{\downarrow}]$ of any principal downset $x^{\downarrow} \subseteq P_2$ is a principal downset in P_1 [4]. Equivalently, there exists a *residual* map $f^r : P_2 \mapsto P_1$ such that

$$\text{(uRES) } f(x) \leq_2 y \text{ iff } x \leq_1 f^r(y)$$

for all $x \in P_1, y \in P_2$.

NL is a logic of one binary operation $\otimes$ on a poset $(P, \leq)$ such that, for any $w \in P$, the maps $\lambda x.x \otimes w$ and $\lambda x.w \otimes x$ from P to P are residuated. Equivalently, the binary operation $\otimes$ admits two *residual* operations $\backslash, /$, satisfying:

$$\text{(bRES) } x \otimes y \leq z \text{ iff } y \leq x \backslash z \text{ iff } x \leq z/y\,,$$

for all $x, y, z \in P$.

It is natural to consider a more general situation. A map $f : P_1 \times \cdots \times P_n \mapsto P$, where $(P_i, \leq_i)$, for $i = 1, \ldots, n$, and $(P, \leq)$ are posets, is said to be *residuated*, if, for any $i = 1, \ldots, n$, the unary maps $\lambda x.f(w_1, \ldots, x : i, \ldots, w_n)$ are residuated, for all $w_1 \in P_1, \ldots, w_n \in P_n$. (Here $x : i$ means that x is the $i-$th argument of f; clearly $w_i \in P_i$ is dropped from the latter list.) Equivalently, the map f admits n residual maps $f^{r,i}$, for $i = 1, \ldots, n$, satisfying:

$$\text{(RES) } f(x_1, \ldots, x_n) \leq z \text{ iff } x_i \leq_i f^{r,i}(x_1, \ldots, z : i, \ldots, x_n)\,,$$

for all $x_1 \in P_1, \ldots, x_n \in P_n, z \in P$, where:

$$f^{r,i} : P_1 \times \cdots \times P : i \times \cdots \times P_n \mapsto P_i\,.$$

Every *identity* map $I(x) = x$ from P to P is residuated, and its residual is the same map. We write $\bar{P}_{(n)}$ for $P_1 \times \cdots \times P_n$. If $f : \bar{P}_{(n)} \mapsto P$ and $g : \bar{Q}_{(m)} \mapsto P_i$ are residuated, then their composition $h : P_1 \times \cdots P_{i-1} \times \bar{Q}_{(m)} \times P_{i+1} \cdots \times P_n \mapsto P$ is residuated, where one sets:

$$h(\ldots, y_1, \ldots, y_m, \ldots) = f(\ldots, g(y_1, \ldots, y_m), \ldots)\,.$$

WARNING. The residuated maps are not closed under a stronger composition operation which from $f, g_1, \ldots, g_k$ yields $h(\bar{x}) = f(g_1(\bar{x}), \ldots, g_k(\bar{x}))$, where $\bar{x}$ stands for $(x_1, \ldots, x_n)$. This composition is considered in recursion theory.

Consequently, posets and residuated maps form a multicategory; posets and unary residuated maps form a category. Notice that an $n-$ary residuated map from $\bar{P}_{(n)}$ to P need not be residuated, if considered as a unary map, defined on the product poset. This can easily be seen, if one notices that an $n-$ary residuated map must be completely additive in each argument, it means:

$$f(\ldots, \bigvee_t x_i^t, \ldots) = \bigvee_t f(\ldots, x_i^t, \ldots)\,,$$

if $\bigvee_t x_i^t$ exists. (If $P_1, \ldots, P_n, P$ are complete lattices, then f is residuated iff it is completely additive in each argument.) Treated as a unary residuated map, it should satisfy a stronger condition: preserve bounds with respect to the product order:

$$f(\bigvee_t (x_1^t, \ldots, x_n^t)) = \bigvee_t f(x_1^t, \ldots, x_n^t)\,.$$

A more concrete example is as follows. Let $(P, \leq)$ be a bounded poset, and let $\otimes$ be a binary residuated map from P^2 to P. We have $\bot \otimes \top = \bot$ and $\top \otimes \bot = \bot$. Then $\otimes^{-1}[\{\bot\}]$ contains the pairs $(\bot, \top), (\top, \bot)$ whose l.u.b. (in the product poset) is $(\top, \top)$. But, in general, $\top \otimes \top \neq \bot$, hence $\otimes^{-1}[\{\bot\}]$ need not be a principal downset. If all universes are complete lattices, then every unary residuated map from the product lattice is an $n-$ary residuated map in the above sense.

If f is a residuated map from $(P, \leq_P)$ to $(Q, \leq_Q)$, then f^r is a residuated map from $(Q, \geq_Q)$ to $(P, \geq_P)$, and f is the residual of f^r. For an $n-$ary residuated map $f : P_1 \times \cdots \times P_n \mapsto Q$, $f^{r,i}$ is a residuated map from $P_1 \times \cdots \times Q^{op} \times \cdots \times P_n$ to P_i^{op}, where P^{op} denotes the poset dual to P; the $i-$th residual of $f^{r,i}$ is f, and the $j-$th residual $(j \neq i)$ is $g(x_1, \ldots, x_n) = f^{r,j}(x_1, \ldots, x_j : i, \ldots, x_i : j, \ldots, x_n)$. Accordingly there is a symmetry between all maps $f, f^{r,1}, \ldots, f^{r,n}$, not explicit in the basic definition. These symmetries will be exploited in section 3.

Residuated maps appear in many areas of mathematics, often defined as Galois connections. *A Galois connection* between posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ is a pair $f : P_1 \mapsto P_2$, $g : P_2 \mapsto P_1$ such that, for all $x \in P_1, y \in P_2$, $x \leq_1 g(y)$ iff $y \leq_2 f(x)$. Clearly, f, g is a Galois connection iff g is the residual of f when $\leq_2$ is replaced by its reversal. In opposition to residuated maps, the first (second) components of Galois connections are not closed under composition (hence residuated maps lead to a more elegant framework [4]).

Residuated maps in mathematics usually act between different universes, like in the classical Galois example: between groups and fields. On the other hand, the logical theory of residuation focused, as a rule, on the one-universe case, and similarly for the algebraic theory. One considers different kinds of *residuation algebras*, e.g. residuated semigroups (groupoids), (nonassociative) residuated lattices, their expansions with unary operations, and so on, together with the corresponding logics; see e.g. [4, 11]. Typically all operations are (unary or binary) operations in the algebra. The situation is similar in linguistic approaches, traditionally developed in connection with type grammars based on different variants of the Lambek calculus.

We provide some examples of residuated maps.

$\mathcal{P}(W)$ is the powerset of W. A residuated map from $\mathcal{P}(V_1) \times \cdots \times \mathcal{P}(V_n)$ to $\mathcal{P}(W)$ can be defined as follows. Let $R \subseteq W \times V_1 \times \cdots \times V_n$. For $(X_1, \ldots, X_n)$, where $X_j \subseteq V_j$, for $j = 1, \ldots, n$, one defines:

$$f_R(X_1, \ldots, X_n) = \{y \in W : (\exists x_1 \in X_1, \ldots, x_n \in X_n) R(y, x_1, \ldots, x_n)\} .$$

f_R is residuated, and its residual maps are:

$$f_R^{r,i}(X_1, \ldots, Y : i, \ldots, X_n) = \{x \in V_i : f_R(X_1, \ldots, \{x\} : i, \ldots, X_n) \subseteq Y\} .$$

For $n = 1$ and $V_1 = W$, f_R is the $\Diamond-$modality determined by the Kripke frame (W, R), $R \subseteq W^2$; see e.g. [3]. Precisely, it is the operation corresponding to $\Diamond$ in the complex algebra of the frame. Analogously, for $V_i = W$, $i = 1, \ldots, n$, f_R corresponds to the $\Diamond$ determined by the multi-modal frame (W, R), $R \subseteq W^{n+1}$. To get the correspondence, the truth definition should be: $y \models \Diamond\varphi$ iff, for some x, $R(y, x)$ and $x \models \varphi$, and similarly for the multi-modal case. If one defines: $\|\varphi\| = \{x \in W : x \models \varphi\}$, then $\Diamond(\|\varphi\|) = \|\Diamond(\varphi)\|$, where the first $\Diamond$ is the operation f_R, and the second one is the corresponding modal connective.

If R is not symmetric, then f_R^r does not equal the $\Box-$modality corresponding to $\Diamond$, namely $\Box(X) = -\Diamond(-X)$. One often writes $\Box^{\downarrow}$ for f_R^r. Modal logics are usually presented with the modal pair $\Diamond, \Box$, but without $\Box^{\downarrow}$. Some exceptions are temporal logics with their residual pairs F, H and P, G, and some substructural modal logics. Let us notice that every normal modal logic which is complete with respect to a class of Kripke frames can conservatively be expanded by adding $\Box^{\downarrow}$, the residual of $\Diamond$. Such expansions inherit basic properties of normal modal logics, and they can be studied by certain methods of substructural logics.

Dynamic logics make the connection between R and $\Diamond$ explicit; one writes $\langle R \rangle$ for the $\Diamond$ determined by R, and $[R]$ for its De Morgan dual; instead of R one writes a program term interpreted as R.

A greater flexibility can be attained by treating $\Diamond$ as a binary map from $(\mathcal{P}(W^2)) \times \mathcal{P}(W)$ to $\mathcal{P}(W)$: $\Diamond(R, X) = \{y \in W : (\exists x \in X) R(y, x)\}$. In this setting $\Diamond = f_S$, where $S \subseteq W \times W^2 \times W$ consists of all tuples $(y, (y, x), x)$ such that $x, y \in W$. Notice that S is a *logical* relation, since it is invariant under permutations of W.

Consequently the binary $\Diamond$ is residuated. We have:

$$\Diamond^{r,2}(R, X) = [R]^{\downarrow}(X) = [R^{\smile}](X) = \{x \in W : \Diamond(R, \{x\}) \subseteq X\}.$$

The other residual:

$$\Diamond^{r,1}(X, Y) = \{(x, y) \in W^2 : \Diamond(\{(x, y)\}, Y) \subseteq X\} =$$

$$= \{(x, y) \in W^2 : x \in X \sqcup y \notin Y\}$$

yields the greatest relation R such that $\Diamond(R, Y) \subseteq X$. It is not a standard operation in dynamic logics, but it may be quite useful. If φ, ψ are formulas,

$\Diamond^{r,1}(\|\varphi\|,\|\psi\|)$ is interpreted as the largest (nondeterministic) program R such that, for any input satisfying the pre-condition $\neg\varphi$, every outcome of R satisfies the post-condition $\neg\psi$. Besides known laws of dynamic logic, in the extended language one can express new laws, e.g.:

$$\Diamond^{r,1}(\|\varphi\wedge\psi\|,\|\chi\|)=\Diamond^{r,1}(\|\varphi\|,\|\chi\|)\cap\Diamond^{r,1}(\|\psi\|,\|\chi\|)\,,$$

$$\Diamond^{r,1}(\|\varphi\|,\|\psi\vee\chi\|)=\Diamond^{r,1}(\|\varphi\|,\|\psi\|)\cap\Diamond^{r,1}(\|\varphi\|,\|\chi\|)\,.$$

(In general, if f is residuated, then $f^{r,i}$ preserves all existing meets in the $i-$th argument, and sends the existing joins to the corresponding meets in any other argument.) Clearly the binary $\Diamond$ with its residuals is an example of a multi-sorted residuation triple. They are logical operations in the above sense.

Other examples of logical multi-sorted residuated maps are the relative product map $\circ:\mathcal{P}(U\times V)\times\mathcal{P}(V\times W)\mapsto\mathcal{P}(U\times W)$, the Cartesian product map $\times:\mathcal{P}(V)\times\mathcal{P}(W)\mapsto\mathcal{P}(V\times W)$, and the disjoint union map $\uplus:\mathcal{P}(V)\times\mathcal{P}(W)\mapsto\mathcal{P}(V\uplus W)$.

Given any map $g: V_1\times\cdots\times V_n\mapsto W$, by $R(g)$ we denote the relation: $R(g)(y,x_1,\ldots,x_n)$ iff $y=g(x_1,\ldots,x_n)$ (the graph of g). The residuated map $f_{R(g)}$ will be denoted by p_g. This construction appears in numerous applications. We mention some examples connected with linguistics.

A standard interpretation of **NL** involves binary *skeletal* trees, i.e. trees whose leaves but no other nodes are labeled by certain symbols. Clearly skeletal trees can be represented as bracketed strings over some set of symbols. Let $\Sigma=\{a,b\}$. Then $[a,[b,a]]$ represents the tree on Figure 1.

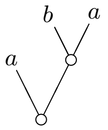

Fig. 1. A binary skeletal tree.

The formulas of **NL** are interpreted as sets of skeletal trees (over an alphabet Σ), and the product connective $\otimes$ is interpreted as p_*, where $*$ is the concatenation of skeletal trees: $t_1*t_2=[t_1,t_2]$.

If skeletal trees are replaced with labeled trees whose internal nodes are labeled by category symbols, then instead of one operation $*$ one must use a family of operations $*_A$, one for each category symbol A. One defines: $t_1*_At_2=[t_1,t_2]_A$. Often binary operations are not sufficient; one needs $n-$ary operations for $n=1,2,3,\ldots$. For instance, a ternary operation o_A sends (t_1,t_2,t_3) to $[t_1,t_2,t_3]_A$. This leads to the formalism of Generalized Lambek Calculus.

In the above setting we admit that an $n-$ary operation is defined on all possible $n-$tuples of trees. As a result, we generate a huge universe of trees, many of them being completely useless for syntactic analysis. This overgeneration can be eliminated, if one restricts the application of an operation to those tuples which satisfy additional constraints. To formalize this idea we might admit partial operations, which would essentially complicate the algebraic and logical details.

Here we describe another option, involving multi-sorted operations. Let G be a context-free grammar (CFG) in a normal form: every production rule of G is of the form $A \to B_1, \ldots, B_n$, where $n \geq 1$ and A, B_i are nonterminals, or $A \to a$, where A is a nonterminal, a is a terminal symbol from Σ. The rules of the first form are called *tree rules*, and those of the second form are called *lexical rules*.

Let T_A denote the set of all labeled trees whose root is labeled by A. With any tree rule r we associate an operation o_r; if r is $A \to B_1, \ldots, B_n$, then $o_r : T_{B_1} \times \cdots T_{B_n} \mapsto T_A$ is defined as follows: $o_r(t_1, \ldots, t_n) = [t_1, \ldots, t_n]_A$.

L_A denotes the set of all *lexical* trees $[a]_A$ such that $A \to a$ is a lexical rule. D_A denotes the set of all (complete) derivation trees of G whose root is labeled by A.

The sets T_A with the operations o_r form a multi-sorted algebra, and the sets $D_A \subseteq T_A$ with the same operations (naturally restricted) form a subalgebra of this algebra; it is the subalgebra generated by the lexical trees. Precise definitions of these notions will be given in section 3. Speaking less formally, if one starts from lexical trees and applies operations o_r, then the generated trees are precisely the derivation trees of G. For instance, let the rules of G be $r_1 : S \to S, B$; $r_2 : S \to A, B$; $A \to a$; $B \to b$. Figure 2 shows a tree in D_S.

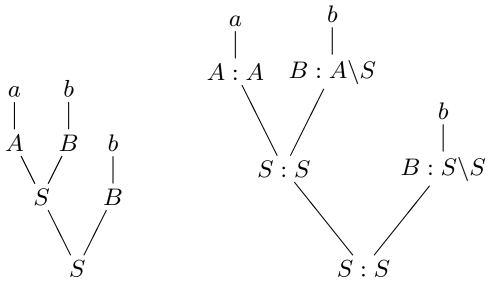

Fig. 2. The tree $o_{r_1}(o_{r_2}([a]_A, [b]_B), [b]_B)$ and its typed version.

A type grammar G' equivalent to G assigns: $a : A$, $b : A\backslash S$, $S\backslash S$. To attain a full coincidence of derivation trees we assign types to lexical trees: $[a]_A : A$, $[b]_B : A\backslash S$, $S\backslash S$. Then, **NL** (actually the pure reduction calculus **AB**) yields essentially the derivation trees of G; see Figure 2. The label $A : \alpha$ means that the tree with root A is of type α.

The grammar G' should be modified to be fully compatible with the multi-sorted framework. One should take $[b]_B : A\backslash_2 S$, $S\backslash_1 S$. Then, in the algebra of sets of trees one interprets o_{r_i} as the operation $p_i = p_{o_{r_i}}$, and $\backslash_i$ is interpreted as the 2-nd residual of p_i. The typing of non-lexical subtrees of the above tree agrees with basic reduction laws $p_i(X, p_i^{r,2}(X,Y)) \subseteq Y$, which follow from (RES).

The above example illustrates one of many possible applications of multi-sorted operations in language description: a type grammar describes syntactic trees generated by a CFG. The CFG may provide a preliminary syntactic analysis, while the type grammar gives a more subtle account, or the grammars may focus on different features (like in applications of product pregroups [24, 10]).

Another obvious option is a multi-level grammar, which handles both the syntactic and the semantic level; a two-sorted meaning map m sends syntactic trees into semantic descriptions (m need not be residuated, but the powerset map p_m certainly is). We can also imagine a joint description of strings (unstructured expressions) and trees (structured expressions) with a forgetting map from structures to strings; also expressions from two different languages with translation maps. Other examples will be mentioned in section 4.

3 Multi-sorted residuation algebras

According to [7], *a residuated algebra* (RA) is a poset $(A, \leq)$ with a family F of residuated operations on A; each $n-$ary operation $f \in F$ admits n residual operations $f^{r,i}$, $1 \leq i \leq n$. (In [7], $o, o/i$ are used instead of $f, f^{r,i}$.) One also considers residuated algebras with lattice operations $\sqcup, \sqcap$ and Boolean negation or Heyting implication. The corresponding logics are Generalized Lambek Calculus and its extensions. The term 'residuated algebra' was coined after 'residuated lattice', used in the literature on substructural logics. Here we prefer 'residuation algebra', since the operations (not the algebra) are residuated; also 'residuated lattice' seems (even more) unlucky, since the residuals are not directly related to the lattice operations.

A multi-sorted residuation algebra (mRA) is a family $\{A_s\}_{s \in S}$ of ordered algebras with a family F of residuated maps; each map $f \in F$ is assigned a unique type $s_1, \ldots, s_n \to s$, where $s_i, s \in S$, and $f : A_{s_1} \times \cdots \times A_{s_n} \mapsto A_s$. S is the set of *sorts*. So a map f of type $s_1, \ldots, s_n \to s$ admits n residual maps:

$$f^{r,i} : A_{s_1} \times \cdots \times A_s : i \times \cdots \times A_{s_n} \mapsto A_{s_i} \,.$$

The ordered algebras A_s are always posets, but some of them can also admit semilattice, lattice, boolean or Heyting operations. A mRA is often denoted $\mathcal{A} = (\{A_s\}_{s \in S}, F)$ (we also write $F_{\mathcal{A}}$ for F).

A subalgebra of $\mathcal{A}$ is a family $\{B_s\}_{s \in S}$ such that $B_s \subseteq A_s$ and this family is closed under the operations from $F_{\mathcal{A}}$ and their residuals (dropping residuals, one obtains a standard notion of a subalgebra of a multi-sorted algebra). Clearly a subalgebra of a mRA is also a mRA with appropriately restricted operations.

Two mRAs $\mathcal{A}$, $\mathcal{B}$ are said to be *similar*, if they have the same set of sorts S, $F_{\mathcal{A}} = \{f_i\}_{i \in I}$, $F_{\mathcal{B}} = \{g_i\}_{i \in I}$, and f_i, g_i are of the same type, for any $i \in I$; we also assume that A_s, B_s are of the same type, for any $s \in S$ (it means: both are posets or lattices, semilattices, etc.). *A homomorphism* from $\mathcal{A}$ to $\mathcal{B}$, which are similar, is a family $\{h_s\}_{s \in S}$ such that $h_s : A_s \mapsto B_s$ is a homomorphism of ordered algebras, and the following equations hold, for any f_j of type $s_1, \ldots, s_n \to s$ and all $1 \leq i \leq n$:

$$\text{(HOM1) } h_s(f_j(a_1, \ldots, a_n)) = g_j(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))\,,$$

$$\text{(HOM2) } h_{s_i}(f_j^{r,i}(a_1, \ldots, a_n)) = g_j^{r,i}(h_{s_1}(a_1), \ldots, h_s(a_i) : i, \ldots, h_{s_n}(a_n))\,.$$

We assume (HOM1) for all $a_1 \in A_{s_1}, \ldots, a_n \in A_{s_n}$, and (HOM2) for all $a_k \in A_{s_k}$, for $k \neq i$, and $a_i \in A_s$. *An embedding* is a homomorphism $\{h_s\}_{s \in S}$ such that every h_s is an embedding, it means: $a \leq_{A_s} b$ iff $h_s(a) \leq_{B_s} h_s(b)$, for all $a, b \in A_s$.

Standard examples of mRAs are *complex algebras* of multi-sorted frames $(\{V_s\}_{s \in S}, \mathcal{R})$ such that every V_s is a set, and $\mathcal{R}$ is a family of relations, each $R \in \mathcal{R}$ having a unique type $s_1, \ldots, s_n \to s$, and $R \subseteq V_s \times V_{s_1} \times \cdots \times V_{s_n}$. The given relation R determines a residuated map f_R, as defined in section 2. *The complex* mRA associated with the frame is defined as $(\{\mathcal{P}(V_s)\}_{s \in S}, \{f_R\}_{R \in \mathcal{R}})$. Clearly every $\mathcal{P}(V_s)$ is a boolean algebra of sets, and the ordering on $\mathcal{P}(V_s)$ is inclusion.

If all algebras A_s in $\mathcal{A}$ are of the same type, say posets or distributive lattices, admitting boolean algebras (we only consider these types; see the remarks at the end of this section), then *$\mathcal{A}$ can be embedded in the complex algebra of some multi-sorted frame.* This result generalizes known results on *canonical embeddings* of modal algebras, tracing back to [16, 17]. Closely related results for *gaggles* (restricted to one sort) have been presented in [2]. Below we sketch a proof for many sorts, which seems more uniform than those in [2]: we make use of some order dualities and antitone operators to reduce the case of residual operations to that of basic (additive) operations.

Let $\mathcal{A} = (\{A_s\}_{s \in S}, F)$ be a mRA with all ordered algebras of the same type. We define the *canonical* frame $\mathcal{A}^c$ as follows. V_s is defined as the set of:

- all proper upsets of A_s, if A_s is a poset,
- all prime filters of A_s, if A_s is a distributive lattice (a boolean algebra).

A proper upset is a nonempty upset, different from A_s. A filter is an upset closed under meets, and a proper filter is a filter being a proper upset. A prime filter of A_s is a proper filter $X \subseteq A_s$ such that, for all $a, b \in A_s$, $a \sqcup b \in X$ entails $a \in X$ or $b \in X$. The prime filters of a boolean algebra are precisely its ultrafilters.

Let $g \in F$ be of type $s_1, \ldots, s_n \to s$. The relation $R[g] \subseteq V_s \times V_{s_1} \times \cdots \times V_{s_n}$ is defined as follows:

$$\text{(CAN1) } R[g](Y, X_1, \ldots, X_n) \text{ iff } p_g(X_1, \ldots, X_n) \subseteq Y\,,$$

where p_g is defined as in section 2. The complex mRA of $\mathcal{A}^c$ is defined as above.

The canonical embedding $\{h_s\}_{s \in S}$ is defined as follows:

$$\text{(CAN2) } h_s(a) = \{X \in V_s : a \in X\}\,.$$

Clearly $h_s : A_s \mapsto \mathcal{P}(V_s)$. Also $a \leq_{A_s} b$ iff $h_s(a) \subseteq h_s(b)$. The implication $(\Rightarrow)$ holds, since all elements of V_s are upsets. The implication $(\Leftarrow)$ is obvious for posets. If A_s is a distributive lattice and $a \leq_{A_s} b$ is not true, then there exists a prime filter $X \subseteq A_s$ such that $a \in X$, $b \notin X$.

h_s preserves lattice operations. If A_s is a distributive lattice and X is a prime filter of A_s, then $a \sqcap b \in X$ iff $a \in X$ and $b \in X$, and $a \sqcup b \in X$ iff $a \in X$ or $b \in X$, so $h_s(a \sqcap b) = h_s(a) \cap h_s(b)$ and $h_s(a \sqcup b) = h_s(a) \cup h_s(b)$. If A_s is a boolean algebra and $X \subseteq A_s$ is an ultrafilter, then $-a \in X$ iff $a \notin X$, so $h_s(-a) = -h_s(a)$.

We show that $\{h_s\}_{s \in S}$ preserves the operations in F and their residuals. Let $g \in F$ be of type $s_1, \ldots, s_n \to s$. We prove:

$$h_s(g(a_1, \ldots, a_n)) = f_{R[g]}(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))\,. \tag{1}$$

The proof of (1) is correct for any g which in every argument preserves all finite joins, including the empty join, if it exists (this means: $g(a_1, \ldots, a_n) = \bot$ whenever $a_i = \bot$, for some i). For the case of posets, one only assumes that g is isotone in each argument and preserves the empty join.

We show $\subseteq$; the converse inclusion is easy. Let $X \in h_s(g(a_1, \ldots, a_n))$, hence $g(a_1, \ldots, a_n) \in X$. Since g is isotone in each argument, and X is an upset, then $p_g((a_1)^\uparrow, \ldots, (a_n)^\uparrow) \subseteq X$. One shows that for any $1 \leq i \leq n$: (EXT) there exist $X_1 \in h_{s_1}(a_1), \ldots, X_i \in h_{s_i}(a_i)$ such that $p_g(X_1, \ldots, X_i, (a_{i+1})^\uparrow, \ldots, (a_n)^\uparrow) \subseteq X$. Consequently, for $i = n$, one obtains $R[g](X, X_1, \ldots, X_n)$, for some $X_i \in h_{s_i}(a_i)$, $i = 1, \ldots, n$, which yields $X \in f_{R[g]}(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$.

(EXT) is proved by induction on i. Assume that it holds for all $j < i$. If A_{s_i} is a poset, we set $X_i = (a_i)^\uparrow$; it is proper, since $a_i \neq \bot$; otherwise $g(a_1, \ldots, a_n) = \bot$, hence $\bot \in X$, which is impossible. Let A_{s_i} be a distributive lattice. If $p_g(X_1, \ldots, Y : i, (a_{i+1})^\uparrow, \ldots, (a_n)^\uparrow) \subseteq X$ holds for $Y = A_{s_i}$, then X_i can be any prime filter containing a_i (it exists, since $a_i \neq \bot$). Otherwise one considers the family $\mathcal{F}$ of all proper filters $Y \subseteq A_{s_i}$ such that $p_g(X_1, \ldots, Y : i, (a_{i+1})^\uparrow, \ldots) \subseteq X$. $\mathcal{F}$ is nonempty, since $(a_i)^\uparrow \in \mathcal{F}$. By the maximality principle, $\mathcal{F}$ has a maximal element Z. One shows that Z is prime and sets $X_i = Z$.

Suppose that Z is not prime. Then there exist $a, b \notin Z$ such that $a \sqcup b \in Z$. One defines $Z_a = \{y \in A_{s_i} : (\exists x \in Z) a \sqcap x \leq y\}$, and similarly for Z_b. Z_a, Z_b are proper filters (we have $b \notin Z_a$ and $a \notin Z_b$) containing Z and different from Z (we have $a \in Z_a$ and $b \in Z_b$). Accordingly $Z_a, Z_b \notin \mathcal{F}$. Then, for some $x_1 \in X_1, \ldots, x_{i-1} \in X_{i-1}, z_1 \in Z$, $g(x_1, \ldots, x_{i-1}, a \sqcap z_1, a_{i+1}, \ldots, a_n) \notin X$ and, for some $y_1 \in X_1, \ldots, y_{i-1} \in X_{i-1}, z_2 \in Z$, $g(y_1, \ldots, y_{i-1}, b \sqcap z_2, a_{i+1}, \ldots, a_n) \notin X$. Define $u_j = x_j \sqcap y_j$, $z = z_1 \sqcap z_2$. We have $g(u_1, \ldots, u_{i-1}, a \sqcap z, a_{i+1}, \ldots, a_n) \notin X$ and $g(u_1, \ldots, u_{i-1}, b \sqcap z, a_{i+1}, \ldots, a_n) \notin X$. Since X is prime, the join of the latter elements does not belong to X, but it equals $g(u_1, \ldots, u_{i-1}, (a \sqcup b) \sqcap z, a_{i+1}, \ldots, a_n)$. This is impossible, since $Z \in \mathcal{F}$.

For $g^{r,i} : A_{s_1} \times \cdots \times A_s : i \times \cdots \times A_{s_n} \mapsto A_{s_i}$, we prove:

$$h_{s_i}(g^{r,i}(a_1, \ldots, a_n)) = f^{r,i}_{R[g]}(h_{s_1}(a_1), \ldots, h_s(a_i), \ldots, h_{s_n}(a_n))\,. \tag{2}$$

While the proof of (1) follows routine lines, tracing back to [16] (also see [2]), our proof of (2) is different. We reduce (2) to (1) by applying some dualities.

By $A_s^{op}, A_{s_i}^{op}$ we denote the algebras dual to A_s, A_{s_i}, respectively; the ordering in the dual algebra is the reversal of the ordering in the initial algebra. Thus, one interchanges $\bot$ with $\top$, and $\sqcup$ with $\sqcap$ in lattices.

By g' we denote the mapping from $A_{s_1} \times \cdots \times A_s^{op} : i \times \cdots \times A_{s_n}$ to $A_{s_i}^{op}$ which equals $g^{r,i}$ as a function. Since $g^{r,i}$ respects arbitrary meets in the $i-$th argument and turns joins into meets in the other arguments, then g' respects finite joins in each argument. So g' satisfies the requirements needed in the proof of (1). We, however, must replace $\mathcal{A}$ by $\mathcal{A}'$ in which A_s, A_{s_i} are replaced by $A_s^{op}, A_{s_i}^{op}$, respectively. Precisely, we assume that now $s_1, \ldots, s_n, s$ are different sorts, if even they are not different in $\mathcal{A}$. Actually our argument only depends on the fixed operations $g, g^{r,i}$, not on the whole frame $\mathcal{A}$, so we may modify it for the purposes of this argument.

In the canonical frame $(\mathcal{A}')^c$, $(V_s)'$ consists of all proper upsets of A_s^{op}, hence all proper downsets of A_s, if A_s is a poset, and all prime filters of A_s^{op}, hence all prime ideals of A_s, if A_s is a distributive lattice, and similarly for $(V_{s_i})'$. The homomorphism $\{k_s\}_{s \in S}$ is defined as $\{h_s\}_{s \in S}$ except that $\mathcal{A}$ is replaced by $\mathcal{A}'$, and similarly for the canonical frame. (1) yields:

$$k_{s_i}(g'(a_1, \ldots, a_n)) = f_{R[g']}(k_{s_1}(a_1), \ldots, k_s(a_i), \ldots, k_{s_n}(a_n))\,, \tag{3}$$

where $f_{R[g']}$ is defined in the complex algebra of $(\mathcal{A}')^c$.

For any $t \in S$, $X \subseteq A_t$, we denote $-X = A_t - X$. For $U \subseteq V_t$, we denote $\sim_{V_t} U = V_t - U$, $U^{\sim} = \{-X : X \in U\}$. We define the auxiliary operations: ${}^*_t(-) : \mathcal{P}((V_t)') \mapsto \mathcal{P}(V_t)$ and $(-)^*_t : \mathcal{P}(V_t) \mapsto \mathcal{P}((V_t)')$, for $t = s$ and $t = s_i$:

$${}^*_t(U) = \sim_{V_t} (U^{\sim})\,,\ (V)^*_t = \sim_{(V_t)'} (V^{\sim})\,, \tag{4}$$

for $U \subseteq (V_t)'$, $V \subseteq V_t$. We write ${}^*U$, V^* for ${}^*_t(U)$, $(V)^*_t$.

One easily shows ${}^*U = (\sim_{(V_t)'} U)^{\sim}$ and $V^* = (\sim_{V_t} V)^{\sim}$. The operations ${}^*(-)$ and $(-)^*$ are antitone and $({}^*U)^* = U$, ${}^*(V^*) = V$. Also, for $t = s$ and $t = s_i$, we have $h_t(a) = {}^*(k_t(a))$, for any $a \in A_t$. For $t = s_j$, $j \neq i$, we have $k_t = h_t$. Since g and g' are equal as functions, then (3) yields:

$$h_{s_i}(g(a_1, \ldots, a_n)) = {}^*(f_{R[g']}(h_{s_1}(a_1), \ldots, (h_s(a_i))^*, \ldots, h_s(a_n)))\,. \tag{5}$$

To prove (2) it suffices to show:

$${}^*(f_{R[g']}(V_1, \ldots, (V_i)^*, \ldots, V_n)) = f^{r,i}_{R[g]}(V_1, \ldots, V_n)\,, \tag{6}$$

for all $V_1 \subseteq V_{s_1}, \ldots, V_i \subseteq V_s, \ldots, V_n \subseteq V_{s_n}$.

One proves (6) by simple computations, using: $X \in {}^*U$ iff $(-X) \notin U$, for all $X \in V_t$, $U \subseteq (V_t)'$ and $X \in V^*$ iff $(-X) \notin V$, for all $X \in (V_t)'$, $V \subseteq V_t$. The following formulas are equivalent.

1. $X \in {}^*(f_{R[g']}(V_1, \ldots, (V_i)^*, \ldots, V_n))$,
2. $(-X) \notin f_{R[g']}(V_1, \ldots, (V_i)^*, \ldots, V_n)$,

3. $\neg R[g'](-X, X_1, \ldots, X_n)$, for all $X_j \in V_j$, $(j \neq i)$, and $X_i \in (V_i)^*$,
4. $\neg R[g'](-X, X_1, \ldots, X_n)$, for all $X_j \in V_j$, $(j \neq i)$, $(-X_i) \notin V_i$,
5. for all $X_j \in V_j$, $(j \neq i)$, $X_i \in (V_s)'$, if $-X_i \notin V_i$ then $\neg R[g'](-X, X_1, \ldots, X_n)$,
6. for all $X_j \in V_j$, $(j \neq i)$, $X_i \in (V_s)'$, if $R[g'](-X, X_1, \ldots, X_n)$ then $(-X_i) \in V_i$,
7. for all $X_j \in V_j$, $(j \neq i)$, $Y_i \in V_s$, if $R[g'](-X, X_1, \ldots, -Y_i, \ldots, X_n)$ then $Y_i \in V_i$,
8. $X \in f^{r,i}_{R[g]}(V_1, \ldots, V_n)$.

For the equivalence of formulas 7 and 8, we need further equivalences. The equivalences of formulas 2-3 and 5-6 below use the fact that, if Y is an upset of a poset $(A, \leq)$ and $a \in A$, then $a \in Y$ iff, for all $b \in A$, if $a \leq b$ then $b \in Y$.

1. $R[g'](-X, X_1, \ldots, -Y_i, \ldots, X_n)$,
2. $p_{g'}(X_1, \ldots, -Y_i, \ldots, X_n) \subseteq -X$,
3. for all $a_j \in X_j$, $(j \neq i)$, $a_i \in A_s, b \in A_{s_i}$, if $a_i \notin Y_i$ and $g^{r,i}(a_1, \ldots, a_n) \leq_{A^{op}_{s_i}} b$ then $b \notin X$,
4. for all $a_j \in X_j$, $(j \neq i)$, $a_i \in A_s, b \in A_{s_i}$, if $a_i \notin Y_i$ and $b \leq_{A_{s_i}} g^{r,i}(a_1, \ldots, a_n)$ then $b \notin X$,
5. for all $a_j \in X_j$, $(j \neq i)$, $a_i \in A_s, b \in A_{s_i}$, if $b \in X$ and $g(a_1, \ldots, b : i, \ldots, a_n) \leq_{A_s} a_i$ then $a_i \in Y_i$,
6. $p_g(X_1, \ldots, X : i, \ldots, X_n) \subseteq Y_i$,
7. $R[g](Y_i, X_1, \ldots, X : i, \ldots, X_n)$.

The proof is finished. As we point out in section 4, the embedding results imply some basic completeness theorems and conservation results for multi-sorted substructural logics. Even for the one-sort case, the above proof brings something new. Even for a basic map $g : A^n \mapsto A$, hence also $g^{r,i} : A^n \mapsto A$, the second part of the proof introduces $g' : A \times \cdots \times A^{op} : i \times \cdots A \mapsto A^{op}$, which is a multi-sorted map. This shows that multi-sorted algebras can be useful for studying standard algebras.

The canonical embedding h preserves $\bot, \top$; we have $h_s(\bot) = \emptyset$ and $h_s(\top) = V_s$. As shown in [21], it also preserves units for binary operations and some non-classical negations; the complex algebra inherits such properties of basic operations as associativity and commutativity (but not idempotence) and preserves the equations of linear logics. These results have been adapted for symmetric residuation algebras (with one sort, but the proof also works for many sorts) in [22], using the * operators on $\mathcal{A}^c$, after [8].

At this moment, the author does not know whether the embedding theorem can be obtained for mRAs in which different A_s can have different types, e.g. some of them are posets, and some others are distributive lattices. The proof of (EXT) (see the proof of (1)) uses the fact that A_s is a lattice whenever A_{s_i} is a lattice (so the converse is needed in the proof of (2)). Obviously, the distributive law cannot be easily avoided; non-distributive lattices cannot be embedded in the complete lattices of sets.

4 Multi-sorted residuation logics

Generalized Lambek Calculus (**GL**) is a logic of RAs. Formulas are formed out of variables by means of operation symbols (connectives) $o, o^{r,i}$ $(1 \leq i \leq n$, if o is $n-$ary). The formal language contains a finite number of operation symbols. These operation symbols are *multiplicative* (or: intensional, according to a different tradition). One can also admit *additive* (or: extensional) symbols $\sqcup, \sqcap$, interpreted as lattice operations, and additive constants $\bot, \top$.

The algebraic form of the multiplicative **GL** admits sequents of the form $A \Rightarrow B$ such that A, B are formulas. The only axioms are

$$\text{(Id) } A \Rightarrow A\,,$$

and the inference rules strictly correspond to the residuation laws (RES): (R-RES) from $o(A_1, \ldots, A_n) \Rightarrow B$ infer $A_i \Rightarrow o^{r,i}(A_1, \ldots, B : i, \ldots, A_n)$, and conversely, (1-CUT) from $A \Rightarrow B$ and $B \Rightarrow C$ infer $A \Rightarrow C$.

An equivalent Gentzen-style system admits sequents of the form $X \Rightarrow A$ such that A is a formula, and X is a formula structure (tree). A formula structure is a formula or an expression of the form $(X_1, \ldots, X_n)_o$ such that each X_i is a formula structure. Here $(-)_o$ is the structural operation symbol corresponding to the $n-$ary multiplicative symbol o.

The axioms are (Id) and (optionally):

$$(\bot \Rightarrow)\ X[\bot] \Rightarrow A \quad (\Rightarrow \top)\ X \Rightarrow \top$$

and the inference rules are:

$$(o \Rightarrow)\ \frac{X[(A_1, \ldots, A_n)_o] \Rightarrow A}{X[o(A_1, \ldots, A_n)] \Rightarrow A} \quad (\Rightarrow o)\ \frac{X_1 \Rightarrow A_1; \ldots; X_n \Rightarrow A_n}{(X_1, \ldots, X_n)_o \Rightarrow o(A_1, \ldots, A_n)}$$

$$(o^{r,i} \Rightarrow)\ \frac{X[A_i] \Rightarrow B;\ (Y_j \Rightarrow A_j)_{j \neq i}}{X[(Y_1, \ldots, o^{r,i}(A_1, \ldots, A_n), \ldots, Y_n)_o] \Rightarrow B}$$

$$(\Rightarrow o^{r,i})\ \frac{(A_1, \ldots, X : i, \ldots, A_n)_o \Rightarrow A_i}{X \Rightarrow o^{r,i}(A_1, \ldots, A_n)}$$

$$(\sqcup \Rightarrow)\ \frac{X[A] \Rightarrow C;\ X[B] \Rightarrow C}{X[A \sqcup B] \Rightarrow C} \quad (\Rightarrow \sqcup)\ \frac{X \Rightarrow A_i}{X \Rightarrow A_1 \sqcup A_2}$$

$$(\sqcap \Rightarrow)\ \frac{X[A_i] \Rightarrow B}{X[A_1 \sqcap A_2] \Rightarrow B} \quad (\Rightarrow \sqcap)\ \frac{X \Rightarrow A;\ X \Rightarrow B}{X \Rightarrow A \sqcap B}$$

$$\text{(CUT)}\ \frac{X[A] \Rightarrow B;\ Y \Rightarrow A}{X[Y] \Rightarrow B}$$

One can also admit constants, treated as nullary operation symbols; they do not possess residuals. For a constant o, one admits rules $(o \Rightarrow)$, $(\Rightarrow o)$ for $n = 0$ (the second one is an axiom):

$$(o \Rightarrow_0)\ \frac{X[()_o] \Rightarrow B}{X[o] \Rightarrow B}\ (\Rightarrow_0 o)\ ()_o \Rightarrow o\,.$$

If a constant has to play a special role, then one needs additional axioms or rules. That 1 is the unit of o (binary) can be axiomatized by means of the following structural rules and their reversals:

$$(1')\ \frac{X[Y] \Rightarrow A}{X[(()_1, Y)_o] \Rightarrow A}\ (1'')\ \frac{X[Y] \Rightarrow A}{X[(Y, ()_1)_o] \Rightarrow A}\,.$$

The above system with additives has been studied in [7] and called there Full Generalized Lambek Calculus (**FGL**). Here we consider its multi-sorted version, called Multi-Sorted Full Generalized Lambek Calculus or, simply, Multi-Sorted Full Lambek Calculus (**mFL**). Its multiplicative fragment is referred to as Multi-Sorted Lambek Calculus (**mL**).

We fix a nonempty set S whose elements are called sorts. Each variable is assigned a unique sort; we write $p : s$. One admits a nonempty set $\mathcal{O}$ whose elements are called operation symbols. Each symbol $o \in \mathcal{O}$ is assigned a unique type of the form $s_1, \ldots, s_n \to s$, where $s_1, \ldots, s_n, s \in S$, $n \geq 1$. If $o : s_1, \ldots, s_n \to s$, then the language also contains operation symbols $o^{r,i}$ $(1 \leq i \leq n)$ such that $o^{r,i} : s_1, \ldots, s : i, \ldots, s_n \to s_i$. One also admits a (possibly empty) set $\mathcal{C}$ whose elements are called constants. Each constant o is assigned a unique sort.

One recursively defines sets F_s, for $s \in S$; the elements of F_s are called formulas of sort s. All variables and constants of sort s belong to F_s; if f is an operation symbol (basic o or residual $o^{r,i}$) of type $s_1, \ldots, s_n \to s$, $(n \geq 0)$, and A_i is a formula of sort s_i, for any $i = 1, \ldots, n$, then $f(A_1, \ldots, A_n)$ is a formula of sort s. In the presence of additives, if $A, B \in F_s$, then $A \sqcup B, A \sqcap B \in F_s$; optionally, also $\bot_s, \top_s \in F_s$. We write $A : s$ for $A \in F_s$.

Each formula of sort s is a formula structure of sort s; if $X_i : s_i$ for $i = 1, \ldots, n$, $(n \geq 0)$, and $o \in \mathcal{O}$ is of type $s_1, \ldots, s_n \to s$, then $(X_1, \ldots, X_n)_o$ is a formula structure of sort s. FS_s denotes the set of formula structures of sort s. We write $X : s$ for $X \in \mathrm{FS}_s$. An expression $X \Rightarrow A$ such that $X \in \mathrm{FS}_s$, $A \in F_s$ is called a sequent of sort s.

The axioms and rules of **mFL** are the same as for **FGL**, but we require that all formulas and sequents must have some sort. Clearly **mFL** is not a single system; we have defined a class of systems, each determined by the particular choice of S and $\mathcal{O}$. Every system from this class admits *cut elimination*, which was first shown for **NL** by Lambek [26].

As an example, we consider a system with one basic binary operation $\otimes$; we write $/$ and $\backslash$ for $\otimes^{r,1}$ and $\otimes^{r,2}$, respectively. We assume $\otimes : s, t \to u$, where s, t, u are different sorts. Hence $/ : u, t \to s$ and $\backslash : s, u \to t$. The following laws of **NL** are provable in **mL** (we use the infix notation).

(NL1) $(A/B) \otimes B \Rightarrow A$, $A \otimes (A\backslash B) \Rightarrow B$,
(NL2) $A \Rightarrow (A \otimes B)/B$, $A \Rightarrow B\backslash(B \otimes A)$,
(NL3) $A \Rightarrow B/(A\backslash B)$, $A \Rightarrow (B/A)\backslash B$,
(NL4) $A/B \Leftrightarrow A/((A/B)\backslash A)$, $A\backslash B \Leftrightarrow (B/(A\backslash B))\backslash B$,

(NL5) $A/B \Leftrightarrow ((A/B) \otimes B)/B$, $A\backslash B \Leftrightarrow A\backslash(A \otimes (A\backslash B))$.

We cannot build formulas of the form $(A \otimes B) \otimes C$, $(A/B)/C$ due to sort restrictions. As a consequence, not all laws of **NL** are provable; e.g. $(((A/B)/C) \otimes C) \otimes B \Rightarrow A$ is not. With new operations one can prove a variant of this law $(((A/B)/'C) \otimes' C) \otimes B \Rightarrow A$ under an appropriate sort assignment. We have $A : u$, $B : t$, $A/B : s$. Assuming $C : v$, $(A/B)/'C : x$, we get $\otimes' : x, v \rightarrow s$, hence $/' : s, v \rightarrow x$. Notice that both the type of $\otimes$ and that of $\otimes'$ consists of three different sorts.

Applying cut elimination, one proves a general theorem: *every sequent provable in* **GL** (hence every sequent provable in **NL**) *results from some sequent provable in* **mL** *in which the type of each operation symbol consists of different sorts (in $s_1, \ldots, s_n \rightarrow s$ all sorts are different), after one has identified all sorts and some operation symbols and variables.* This can be shown by a transformation of a cut-free proof of $X \Rightarrow A$ with all axioms (Id) of the form $p \Rightarrow p$. In the new proof different axioms contain different variables of different sorts; then different premises of any rule have no common variable and no common sort. Every instance of $(\Rightarrow o)$ and $(o^{r,i} \Rightarrow)$ introduces a new operation symbol together with its structural companion and one new sort. Each sequent in the new proof satisfies the above condition. Furthermore, in each sequent, *every residuation family is represented by 0 or 2 symbols (counting structural symbols).* Consequently, every sequent $A \Rightarrow B$ provable in **mL** contains an even number of operation symbols (this also holds for **L**).

Let us look at (NL5). $A \Leftrightarrow B$ means that both $A \Rightarrow B$ and $B \Rightarrow A$ are provable. The $(\Rightarrow)$ part of (NL5) is $A/B \Rightarrow ((A/B) \otimes B)/B$. In **mL** one proves $A/B \Rightarrow ((A/B) \otimes' B)/'B$ (the reader can find appropriate sorts); the symbol $/$ appears twice in the latter sequent, and the second residuation family is represented by $\otimes', /'$. For the $(\Leftarrow)$ part, the appropriate sequent is $((A/'B) \otimes' B)/B \Rightarrow A/B$. This transformation is impossible for **FGL**; e.g $(A \sqcap B)/C \Rightarrow (A/C) \sqcap (B/C)$ contains 3 occurrences of $/$.

S may consist of one sort only, so **GL** is a limit system from the **mL**-class. The above observations show that the apparently opposite case: each operation symbol has a type consisting of different sorts, leads to essentially the same (pure) logic provided that one admits infinite sets $S, \mathcal{O}$.

Some possible applications of **mL** in linguistics have been mentioned in section 2. Another one is subtyping. A 'large' type S (sentence) can be divided in several subtypes, sensitive to Tense, Number, Mode etc.; these subtypes can be represented by different variables (or: constants) of sort S. In **NL** this goal can be accomplished by additional assumptions: $S_i \Rightarrow S$, for any subtype S_i. With additives one can define $S = S_1 \sqcup \cdots \sqcup S_k$ and apply types dependent on features, e.g. 'John' is assigned 'np $\sqcap$ sing, 'boys' type 'np $\sqcap$ pl [19].

By routine methods, one can show that **mL** *is (strongly) complete with respect to mRAs based on posets*, and **mFL** *is (strongly) complete with respect to mRAs based on (optionally: bounded) lattices.* The strong completeness means that, for any set of sequents Φ (treated as nonlogical assumptions), the sequents derivable from Φ in the system are precisely the sequents valid in all models satisfying all sequents from Φ (a model is an algebra with a valuation of variables). In other words, the strong completeness of a system (with respect to a class of algebras) is equivalent to the completeness of the consequence relation of this system (with respect to the class of algebras).

To attain the completeness with respect to mRAs based on distributive lattices, we add the distributive law as a new axiom:

$$\text{(D) } A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)$$

for any formulas A, B, C of the same sort. The resulting system is denoted by **mDFL**. (D) expresses one half of one distributive law; the other half is provable (it holds in every lattice), and the second distributive law is derivable from the first one and basic lattice laws.

This version of **mDFL** does not admit cut elimination. Another version, admitting cut elimination, can be axiomatized like **DFL** in [23] with a structural operation symbol for $\sqcap$ and the corresponding structural rules (an idea originated by J.M. Dunn and G. Mints). We omit somewhat sophisticated details of this approach.

mDFL *is (strongly) complete with respect to mRAs based on distributive lattices.* Soundness is easy, and completeness can be proved, using the Lindenbaum-Tarski algebra (its multi-sorted version). The results from section 3 imply that **mDFL** is strongly complete with respect to the complex mRAs of multi-sorted frames. Soundness is obvious. For completeness, assume that $X \Rightarrow A$ is not derivable from Φ. By the above, there exist a model $(\mathcal{A}, \alpha)$ such that $X \Rightarrow A$ is not true in $(\mathcal{A}, \alpha)$ (it means: $\alpha(X) \leq \alpha(A)$ is not true), but all sequents from Φ are true in $(\mathcal{A}, \alpha)$. Let $\{h_s\}_{s \in S}$ be the canonical embedding of $\mathcal{A}$ in the complex algebra of the frame $\mathcal{A}^c$. The valuation α can be presented as $\{\alpha_s\}_{s \in S}$, where α_s is the restriction of α to variables of sort s (the values of α_s belong to A_s). Then, $\{h_s \circ \alpha_s\}_{s \in S}$ is a valuation in the complex algebra, and the resulting model satisfies all sequents from Φ, but $X \Rightarrow A$ is not true in this model. Ignoring additives, one can prove the same for **mL**. Consequently, the consequence relation of **mDFL** is a conservative extension of the consequence relation of **mL**.

The same holds for Multi-Sorted Boolean Lambek Calculus (**mBL**), which adds to **mDFL** a unary negation (complement) '$-$' and axioms:

$$\text{(N1) } A \sqcap -A \Rightarrow \bot \quad \text{(N2) } \top \Rightarrow A \sqcup -A\,.$$

mBL *is (strongly) complete with respect to boolean mRAs* (all A_s are boolean algebras) as well as *the complex algebras of multi-sorted frames.* (One can also assume that $-A$ can be formed for A of some sorts only.) These results obviously entail the strong completeness of **mBL** and **mDFL** with respect to Kripke frames with standard (classical) clauses for boolean (lattice) operations: $x \models -A$ iff $x \not\models A$, $x \models A \sqcap B$ iff $x \models A$ and $x \models B$, and so on.

In **mBL**, for any operation o, one can define its De Morgan dual. This turns any residuation family to a dual residuation family, which satisfies (RES) with respect to dual orderings; in particular, it yields a faithful interpretation of Moortgat's Symmetric **NL** (without Grishin axioms; see [31]) in **BNL**, i.e. **NL** with boolean operations.

The consequence relation for **L** is undecidable; see [6]. The consequence relation for **mBL** (hence for **mDFL**, **mL**) is decidable (so the pure logics are decidable, too). The proof is similar to that for **DFGL**, **GL** in [9, 7]. One shows Strong Finite Model Property (SFMP): for any finite Φ, if $\Phi \not\vdash X \Rightarrow A$, then there exists a finite multi-sorted model $(\mathcal{A}, \alpha)$ such that all sequents from Φ are but $X \Rightarrow A$ is not true in $(\mathcal{A}, \alpha)$.

The proof of SFMP in [9, 7] uses some interpolation property of sequent systems and a construction of algebras by means of nuclear completions. Different proofs are due to [18] for **BNL** (presented as a Hilbert-style system), by the method of filtration of Kripke frames, and [13] where FEP (see below) has been proved directly for some classes of algebras. Each of them can be adapted for multi-sorted logics.

SFMP yields the decidability of stronger logics: the universal theories of the corresponding classes algebras. Here we refer to a standard translation of substructural logics in first-order language: formulas of these logics correspond to terms and sequents to atomic formulas $t \leq u$. Multi-sorted logics require a multi-sorted first-order language; in particular, $A \Rightarrow B$, where A, B are of sort s, is translated into $t_A \leq_s t_B$, where t_A, t_B are terms of sort s which correspond to A, B.

A Horn formula is a first-order formula of the form $\varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \varphi_{n+1}$, where $n \geq 0$, such that each φ_i is an atomic formula $t \leq_s u$. An open formula is a propositional (boolean) combination of atomic formulas (so Horn formulas are open formulas). A universal sentence results from an open formula by the universal quantification of all variables.

Let $\mathcal{K}$ be a class of algebras. The universal theory of $\mathcal{K}$ is the set of all universal sentences valid in $\mathcal{K}$. The Horn theory of $\mathcal{K}$ is the set of all universally quantified Horn formulas valid in $\mathcal{K}$.

Let a logic $\mathcal{L}$ be strongly complete with respect to $\mathcal{K}$. Then the rules derivable in $\mathcal{L}$ correspond to the Horn formulas belonging to the universal theory of $\mathcal{K}$. Hence the decidability of the universal theory of some class of mRAs (say, boolean residuated groupoids) entails that the problem of derivability of rules in the corresponding logic (here **BNL**) is decidable.

A general, model-theoretic theorem states: *if $\mathcal{K}$ is closed under finite products (admitting the empty product, which yields the trivial algebra), then FMP of the Horn theory of $\mathcal{K}$ entails FMP of the universal theory of $\mathcal{K}$*. For finite languages, FMP of the universal theory of $\mathcal{K}$ is equivalent to Finite Embeddability Property (FEP) of $\mathcal{K}$: every finite, partial subalgebra of an algebra from $\mathcal{K}$ can be embedded in a finite algebra from $\mathcal{K}$. In the literature (see e.g. [11]), the above theorem is formulated for quasi-varieties (which are closed under arbitrary products) in the following form: SFMP for the Horn theory of a quasi-variety $\mathcal{K}$ entails FEP of $\mathcal{K}$, and the proof provides the embedding. Below we sketch another proof, which yields the general result, with arbitrary relation symbols in the language. Also, the usual one-sort algebras can be replaced by multi-sorted algebras. If $\{\mathcal{A}^i\}_{i \in I}$ is a class of similar mRAs, then $\prod_{i \in I} \mathcal{A}^i$ is defined in a natural way: its algebra of sort s equals $\prod_{i \in I} A^i_s$ with point-wise defined relations and lattice (boolean) operations; also the operations in F are defined point-wise. The basic classes of mRAs are closed under arbitrary products (they are multi-sorted quasi-varieties), so this theorem can be applied to them.

Let us sketch the proof. Let $\psi = \forall x_1 \dots x_n \varphi$ be a universal sentence (φ is open). φ is logically equivalent to a CNF- formula $\varphi_1 \wedge \cdots \wedge \varphi_m$, each φ_i being a disjunction of finitely many atomic formulas and negated atomic formulas. So ψ is logically equivalent to the conjunction of ψ_i, $i = 1, \dots, m$, where ψ_i is the universally quantified φ_i. Clearly ψ is valid in an algebra $\mathcal{A}$ iff each ψ_i is valid in $\mathcal{A}$, and the same holds for the validity in $\mathcal{K}$.

Assume that ψ is not valid in $\mathcal{K}$. Then, some sentence ψ_i is not valid. Assuming FMP of the Horn theory, we show that there is a finite algebra in $\mathcal{K}$ such that ψ_i is not true in this algebra. If φ_i consists of negated atomic formulas only, then ψ_i is not true in the trivial algebra, which is finite (an mRA is trivial iff all its algebras A_s are one-element algebras). So assume that φ_i is of the form:

$$\neg\chi_1 \vee \cdots \vee \neg\chi_k \vee \sigma_1 \vee \cdots \vee \sigma_p$$

where $k \geq 0$, $p \geq 1$, and all χ_j, σ_l are atomic. It is logically equivalent to:

$$\chi_1 \wedge \cdots \wedge \chi_k \rightarrow \sigma_1 \vee \cdots \vee \sigma_p\,.$$

Denote $\delta_j = \chi_1 \wedge \cdots \wedge \chi_k \rightarrow \sigma_j$. Since δ_j logically entails φ_i, then δ_j is not valid in $\mathcal{K}$, for $j = 1, \dots, p$. By FMP of the Horn theory, there exists a finite model $(\mathcal{A}^j, \alpha_j)$ over $\mathcal{K}$ which falsifies δ_j. One easily shows that the product model (i.e. the product of all $\mathcal{A}^j$ with the product valuation) falsifies φ_i. Therefore ψ is not true in this product algebra, which finishes the proof.

Since SFMP of our logics is equivalent to FMP of the Horn theories of the corresponding classes of mRAs, then we obtain FMP of the universal theories, which yields their decidability.

The above proof yields: ψ_i is valid in $\mathcal{K}$ iff some δ_j is valid in $\mathcal{K}$. Accordingly, a decision method for the universal theory of $\mathcal{K}$ can be reduced to a decision method for the Horn theory of $\mathcal{K}$ (equivalently: for the consequence relation of the corresponding logic). Some proof-theoretic decision methods for the latter can be designed like for **DFGL** [9, 7], but we skip all details here. We note that a Kripke frame falsifying $\Phi \vdash X \Rightarrow A$ (if it exists) can be found of size at most 2^n, where n is the number of subformulas occurring in this pattern (this was essentially shown in the three proofs of SFMP, mentioned above).

Although **mFL** is decidable, since it admits cut elimination (also FMP holds), the decidability of its consequence relation remains an open problem (even for **FNL**).

The consequence relation of **GL** is polynomial [6]; for the pure **NL** it was earlier shown in [12]. Associative systems **FL**, **DFL** and their various extensions are PSPACE-complete [14]; the proof of PSPACE-hardness (by a reduction of the validity of QBFs to the provability of sequents) essentially relies upon the associative law. Without associativity, by a modification of this proof we can prove the PSPACE-hardness of the consequence relation of **FNL**, **FGL**, **DFGL**, **mFL**, **mDFL** (with at least one binary operation), but the precise complexity of the pure logics is not known. **BNL**, **BGL**, **mBL** are PSPACE-hard, like the modal logic **K**; see e.g. [3].

In [6, 9] it has been shown that the type grammars based on the multiplicative systems and the systems with additives and distribution, also enriched with finitely many assumptions, are equivalent to CFGs.

BGL (i.e. **GL** with boolean operations) is a conservative extension of **K**; it follows from the fact that both **K** and **BGL** are complete with respect to all Kripke frames. (This is obvious, if F contains a unary operation; otherwise, one can reduce an $n-$ary operation to a unary one by fixing some arguments.) A provable formula A of **K** is represented as the provable sequent $\top \Rightarrow A$ of **BGL**; a provable sequent sequent $A \Rightarrow B$ of **BGL** is represented as the provable formula $A \rightarrow B$ of **K**. **mBL** can be treated as a multi-sorted classical modal logic.

Interestingly, some theories based on multi-sorted classical modal logics are paraconsistent: the inconsistency in one sort need not cause the total inconsistency. In algebraic terms, it means that there exist mRAs $\mathcal{A}$ in which some, but not all, algebras A_s are trivial (one-element). Let $A_s = \{a\}$, and let A_t be non-trivial with $\perp_t \in A_t$. Then $f(a) = \perp_t$ is the only residuated map $f : A_s \mapsto A_t$ (notice $a = \perp_s$), and f^r is the constant map: $f^r(x) = a$, for all $x \in A_t$.

There are many natural connections between substructural logics, studied here, and (multi-)modal logics; an early discussion can be found in [1]. Some results, discussed above, have been adapted for one-sort systems admitting special modal axioms (e.g. T, 4, 5) in [29] (FEP, polynomial complexity). This research program seems promising.

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