Lambek Calculus with Classical Logic

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Abstract One of the most natural extensions of the Lambek calculus augments this logic with connectives of classical propositional logic. Actually, the resulting logic can be treated as a classical modal logic with binary modalities. This paper shows several basic properties of the latter logic in two versions: nonassociative and associative (axiom systems, algebras and frames, completeness, decidability, complexity), and some closely related logics. We discuss certain earlier results and add new ones.

1 Introduction

1.1 Overview

The Lambek calculus, introduced by Lambek [31] under the name *Syntactic Calculus*, is a propositional logic which admits three binary connectives $\odot$ (product), $\backslash$ (first residual) and $/$ (second residual); the residuals are also regarded as (substructural) implications and written $\rightarrow$ and $\leftarrow$, respectively. We denote this logic by **L**. It can be axiomatized as a logic of *arrows* $\varphi \Rightarrow \psi$, where φ, ψ are formulas. The axioms are all arrows:

$$\text{(id) } \varphi \Rightarrow \varphi$$

$$\text{(a1) } (\varphi \odot \psi) \odot \chi \Rightarrow \varphi \odot (\psi \odot \chi \quad \text{(a2) } \varphi \odot (\psi \odot \chi) \Rightarrow (\varphi \odot \psi) \odot \chi$$

and the inference rules are as follows:

$$\text{(r1) } \frac{\varphi \odot \psi \Rightarrow \chi}{\psi \Rightarrow \varphi \backslash \chi} \quad \text{(r2) } \frac{\varphi \odot \psi \Rightarrow \chi}{\varphi \Rightarrow \chi / \psi}$$

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$$\text{(cut-1)}\ \frac{\varphi \Rightarrow \psi \quad \psi \Rightarrow \chi}{\varphi \Rightarrow \chi}$$

As usually, the double line in a rule means that it expresses two rules: top-down and bottom-up. This axiomatization follows the algebraic axioms, defining residuated semigroups, which are the algebraic models of **L**. *A residuated semigroup* is an ordered algebra $(A, \odot, \backslash, /, \leq)$ such that $(A, \leq)$ is a poset, $(A, \odot)$ is a semigroup, and $\backslash, /$ are binary operations on A, satisfying *the residuation laws*:

(RES) for all $a, b, c \in A$, $a \odot b \leq c$ iff $b \leq a\backslash c$ iff $a \leq c/a$.

One refers to $\backslash, /$ as *residual operations* for product $\odot$. Lambek [32] also considered a nonassociative version of this logic, nowadays called the nonassociative Lambek calculus and denoted **NL**, which omits (a1), (a2). Its algebraic models are *residuated groupoids*, defined like residuated semigroups except that product need not be associative. In models $\Rightarrow$ is interpreted as $\leq$.

In this paper we consider the extensions of **NL** and **L** by connectives of classical logic: $\neg, \vee, \wedge$ ($\rightarrow$, $\leftrightarrow$ are defined) and constants $\bot$ and $\top$, interpreted as the least and the greatest element in algebras. We denote these logics **NL-CL** and **L-CL**, respectively. They can be axiomatized by adding to **NL** and **L** the following axioms and rules.

$$(\text{a}\wedge)\ \varphi \wedge \psi \Rightarrow \varphi \quad \varphi \wedge \psi \Rightarrow \psi \quad (\text{r}\wedge)\ \frac{\varphi \Rightarrow \psi \quad \varphi \Rightarrow \chi}{\varphi \Rightarrow \psi \wedge \chi}$$

$$(\text{r}\vee)\ \frac{\varphi \Rightarrow \chi \quad \psi \Rightarrow \chi}{\varphi \vee \psi \Rightarrow \chi} \quad (\text{a}\vee)\ \varphi \Rightarrow \varphi \vee \psi \quad \psi \Rightarrow \varphi \vee \psi$$

$$(\text{a}\bot)\ \bot \Rightarrow \varphi \quad (\text{a}\top)\ \varphi \Rightarrow \top$$

$$(\text{D})\ \varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$(\neg.1)\ \varphi \wedge \neg\varphi \Rightarrow \bot \quad (\neg.2)\ \top \Rightarrow \varphi \vee \neg\varphi$$

The first two lines axiomatize $\vee, \wedge$ in lattices. The third line contains axioms for bounds. (D) is a distributive law for $\wedge, \vee$; others are derivable. The last line contains axioms for negation.

Both logics are very natural. The intended models for **L** in linguistics are algebras of languages, i.e. algebras of all subsets of Σ^+ (the set of all nonempty finite strings on Σ). From the point of view of modal logics, a model is based on the complex algebra of a relational frame (W, R), where $R \subseteq W^3$. Dynamic interpretations of **L** lead to relation algebras. These algebras are boolean algebras of sets, hence they make it possible to interpret the connectives of classical logic in a standard way.

One also considers **NL** and **L** with constant 1, interpreted as the unit element for $\odot$. The axioms for 1 are:

$$(\text{ax1})\ 1 \odot \varphi \Leftrightarrow \varphi \quad \varphi \odot 1 \Leftrightarrow \varphi$$

Here and further $\varphi \Leftrightarrow \psi$ means: $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$. The resulting logics are denoted by **NL1** and **L1** and their extensions with classical connectives by **NL1-CL** and **L1-CL**. Notice that **NL1** (resp. **L1**) is not a conservative extension of **NL** (resp.

L). Since $1 \Rightarrow p/p$ is provable in **NL1**, then $(p/p)\backslash p \Rightarrow p$ is so, but the latter is not provable in **NL**. The same example works for **L1** versus **L** and the extensions with classical connectives.

Remark 1 In the literature on linear logics, Lambek connectives $\odot, \backslash, /$ and constants $1, 0$ (see below for 0) are referred to as *multiplicative* and lattice connectives $\vee, \wedge$ and constants $\bot, \top$ as *additive*. In linear logics the notation differs from our, e.g. one writes $\bot$ for our 0, whereas $\top$ and 1 in our sense, and $\oplus$ for our $\vee$. Our notation is similar to that in substructural logics [16].

The researchers in Lambek calculi studied many extensions of NL and **L**. Usually, these extensions differ from ours: they do not use the complete set of classical connectives. Some of them can easily be defined, using the axioms and rules, written above. **L** with $\vee, \wedge$ and (a$\vee$), (r$\vee$), (a$\wedge$), (r$\wedge$) is the logic of lattice-ordered residuated semigroups (we write l.o.r. semigroups, and similarly in other contexts). This logic is called Multiplicative-Additive Lambek Calculus (**MAL**)[1]; see [24], where the acronym is MALC. **MANL** is defined in a similar way. Adding (D) to these logics yields **DMAL** and **DMANL**. They are the logics of distributive l.o.r. (write: d.l.o.r.) semigroups and groupoids, respectively. With 1 and (ax1) one obtains **MAL1**, **MANL1**, **DMAL1**, **DMANL1**. Each logic can be enriched with $\bot, \top$; we use no acronyms for these variants.

In categorial grammars, **NL** and **L** serve as type processing logics. There were considered extensions with several products and the corresponding residuals, with unary modal operators, with $\vee, \wedge$ interpreted in lattices (also distributive lattices), and others. The generative power of categorial grammars based on **L** and **NL** is restricted to context-free languages [39, 9]. **MAL** can generate some mildly context-sensitive languages [23] and similarly for **L-CL**. We briefly discuss categorial grammars at the end of this section.

Substructural logics are often defined as extensions of **L** with $\vee, \wedge$ (interpreted in lattices) and 1 by new axioms and rules. Sequent systems for these logics omit certain structural rules (weakening, contraction, exchange), characteristic of intuitionistic and classical logics; this justifies the name. Full Lambek Calculus **FL** amounts to **MAL1**; it is often regarded as the basic substructural logic. The connectives $\backslash, /$ are treated as nonclassical implications. Assuming the commutative law for $\odot$, the two conditionals $\varphi\backslash\psi$ and ψ/φ collapse in one $\varphi \to \psi$. One defines (substructural) negations: $\sim\varphi = \varphi\backslash 0$, $-\varphi = 0/\varphi$, where 0 is a new constant (interpreted as an arbitrary designated element). These negations are a kind of minimal negation; they collapse in one, if $\odot$ is commutative. Linear logics assume the double negation laws; for Noncommutative **MALL**: $\sim -\varphi \Rightarrow \varphi$, $- \sim \varphi \Rightarrow \varphi$ [1]; the converse arrows are provable. Many nonclassical logics can be presented as axiomatic extensions of **FL** with 0, e.g. many-valued logics, fuzzy logics, constructive logic with strong negation, and others. A thorough discussion of substructural logics and the corresponding algebras can be found in [16].

[1] This name resembles Multiplicative-Additive Linear Logic (**MALL**).

Since $(\varphi\backslash\psi)/\chi \Leftrightarrow \varphi\backslash(\psi/\chi)$ is provable in **L**, then $\sim \varphi/\psi \Leftrightarrow \varphi\backslash -\psi$ is provable in **FL** with 0. This is a substructural counterpart of the contraposition law of intuitionistic logic: $(\psi \to \neg\varphi) \leftrightarrow (\varphi \to \neg\psi)$. Many laws of this kind, which show interplay of negation(s) with Lambek connectives, can be proved in substructural and linear logics. This does not hold for **NL-CL** and **L-CL** in their basic form. It, however, does not mean that the latter are less interesting. In a very natural sense, to be discussed in Section 3, they treat Lambek connectives as modal operators. Roughly $\odot$ is a binary $\Diamond$ and its residuals \, / are similar to $\Box^{\downarrow}$, the backward-looking necessity operator. So **NL-CL** and **L-CL** can be treated as classical modal logics with binary modalities. The present paper focuses on this point of view.

The modal logic interpretation of Lambek calculi was addressed by many authors. In the world of categorial grammars, it was employed by e.g. Morrill [38], Moortgat [36], Moot and Retoré [37], from the perspective of dynamic logics by e.g. van Benthem [46], and in the framework of substructural logics by e.g. Restall [42]. These works, however, usually concern different logics, either weaker than ours, e.g. negation-free, without (D), or incomparable with them, e.g. with several modalities, connected by special axioms.

In [13] a system equivalent to **NL-CL** is denoted by **BFNL** (from: Boolean Full **NL**). The main results are: (1) the strong finite model property (the proof uses algebraic methods, taken from substructural logics), which implies the decidability of provability from fintely many assumptions, (2) the equivalence of categorial grammars based on this logic (also extended by finitely many assumptions) and context-free grammars. In fact, this paper starts from a weaker logic **DFNL**, i.e. our **DMANL**, and the results for **BFNL** are stated at the end with proofs merely outlined. A more general framework, employing residuated algebras with $n-$ary operations, appears in [10].

It is well-known that the provability from (finitely many) assumptions is undecidable for **L**, hence for **L-CL** as well, since the latter is a strongly conservative extension of the former (see Section 2). Therefore, neither **L**, nor **L-CL** possesses the strong finite model property.

Kaminski and Francez [21] study **L-CL** and **NL-CL**, denoted **PL** and **PNL** (from: **L** and **NL** with propositional logic), in the form of Hilbert-style systems. They prove the strong completeness with respect to the corresponding classes of Kripke frames and the strong finite model property for **PNL**, using filtration of Kripke frames.

Several other results are sparse in the literature. The main aim of the present paper is to collect together the most important ones. Nonetheless this paper is not a typical survey. We obtain some new results, write new proofs and simplify (even correct) earlier proofs.

In Section 2 we discuss algebras and Kripke frames, corresponding to **NL-CL**, **L-CL** and some related logics. We prove the strong completeness of these logics w.r.t. (i.e. with respect to) the corresponding classes of algebras and frames. As a consequence, we show that some logics are strongly conservative extensions of others. We also show that these logics are not weakly complete w.r.t. some classes of intended models.

Section 3 presents these logics as Hilbert-style systems (H-systems). The systems from [21] are replaced by others, which makes the analogy with modal logics, in particular: the minimal tense logic $\mathbf{K}_t$, transparent. We add some new modal axioms and study the resulting logics. In particular, cyclic logics are closely related to cyclic linear logics. If one adds axioms $\varphi \odot \psi \to \varphi$ and $\varphi \odot \psi \to \psi$ (corresponding to rules of left weakening in sequent systems), then the resulting logics reduce to classical logic. We show how the standard method of filtration [7] can be adjusted to **NL-CL** and its extensions.

Section 4 concerns decidability and complexity. It is known that **L** is NP-complete [41], whereas even the provability from (finitely many) assumptions in **NL** is PTIME [9]. We write a proof of the undecidability of **L-CL**, which simplifies and corrects the proofs in [28, 29]. **NL-CL** is PSPACE-complete [34]. The provability from (finitely many) assumptions in **NL-CL** is EXPTIME-complete; essentially in [44, 45].

1.2 Categorial grammars

Lambek's intention was to extend the type reduction procedure in categorial grammars, proposed by Ajdukiewicz [2] and modified by Bar-Hillel [4]. Categorial grammars are formal grammars assigning types (categories) to expressions of a language. More precisely, a type lexicon assigns some types to lexical atoms (words), whereas the types of compound expressions are derived by a type reduction procedure (independent of the particular language). The term *categorial grammar* first appeared in Bar-Hillel et al. [5]. This paper, later than and referring to [31], employs reductions of the form[2]:

$$(\mathrm{red}\backslash)\ \alpha, \alpha\backslash\beta \Rightarrow \beta \quad (\mathrm{red}/)\ \alpha/\beta, \beta \Rightarrow \alpha$$

Here α and β are syntactic types; they can be identified with $\backslash, /-$formulas of **L**. An expression $v_1 \ldots v_n$, where $v_1, \ldots, v_n$ are words, is assigned type α, if for some types $\alpha_1, \ldots, \alpha_n$ such that $v_i : \alpha_i$, $i = 1, \ldots, n$, according to the type lexicon the sequence $(\alpha_1, \ldots, \alpha_n)$ reduces to α by finitely many applications of (red\), (red/). For instance, from 'Jane': pn, 'John': pn and 'meets': $(pn\backslash s)/pn$ one derives 'Jane meets John': s by two reductions. This example uses two atomic types: s (sentence) and pn (proper noun).

The arrows (red\) and (red/) are provable in **L** (even **NL**), if one replaces comma with product; e.g. for (red\), apply (r1) (bottom-up) to $\alpha\backslash\beta \Rightarrow \alpha\backslash\beta$. Therefore all derivations based on these reductions can be performed in **L** (even **NL**, if one adds bracketing). **L** yields many new arrows. We list some.

(L1) Type-raising laws: $\alpha \Rightarrow (\beta/\alpha)\backslash\beta$ and $\alpha \Rightarrow \beta/(\alpha\backslash\beta)$
(L2) Composition laws: $\alpha\backslash\beta, \beta\backslash\gamma \Rightarrow \alpha\backslash\gamma$ and $\alpha/\beta, \beta/\gamma \Rightarrow \alpha/\gamma$
(L3) Geach laws[3]: $\alpha\backslash\beta \Rightarrow (\gamma\backslash\alpha)\backslash(\gamma\backslash\beta)$ and $\alpha/\beta \Rightarrow (\alpha/\gamma)/(\beta/\gamma)$

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[2] Reductions in [2] and [4] are more involved, since they employ many-argument types.

[3] Geach [17] was the first who considered categorial grammars with these laws.

Due to these new laws, parsing becomes more flexible. Let us recall Lambek's example. We assign $s/(pn\backslash s)$ (type of subject) to 'she' and $(s/pn)\backslash s$ (type of object) to 'him'. This yields 'she meets John': s by two applications of (red/), but 'she meets him': s needs **L**. In the sequence $s/(pn\backslash s), (pn\backslash s)/pn, (s/pn)\backslash s$ (outer parentheses omitted), one reduces $s/(pn\backslash s), (pn\backslash s)/pn$ to s/pn by (L2), then $s/pn, (s/pn)\backslash s$ to s by (red\). Another way: expand $s/(pn\backslash s)$ to $(s/pn)/((pn\backslash s)/pn)$ by (L3), then use (red/) and (red\). In a categorial grammar based on (red\) and (red/) only, 'he' has to be assigned both types. Accordingly, parsing with **L** enables one to restrict type lexicons and to see logical connections between different types of the same word (expression).

By (L1) every proper noun (type pn) can also be treated as a full noun phrase, both subject (type $s/(pn\backslash s)$) and object (type $(s/pn)\backslash s$). Therefore 'and' in 'Jane and some teacher' can be assigned $(\alpha\backslash\alpha)/\alpha$, where α is the type of subject or object. Another type of 'and' can be $\alpha\backslash(\alpha/\alpha)$, but it is equivalent to the former by the laws of **L**:

(L4) Bi-associativity: $(\alpha\backslash\beta)/\gamma \Leftrightarrow \alpha\backslash(\beta/\gamma)$

(L1) are provable in **NL**; (L2)-(L4) require associativity. Notice that (L4) is needed for 'Jane meets him': s, since the type of 'meets' must be transformed into $pn\backslash(s/pn)$.

In these examples only a few atomic types appear. Lambek [33] elaborates a categorial grammar for a large part of English, which uses 33 atomic types, e.g. π (subject), π_1 (first person singular subject), π_2 (second person singular subject and any plural subject), π_3 (third person singular subject), s (statement), s_1 (statement in present tense), s_2 (statement in past tense), and others. The grammar is based on the calculus of pregroups, but this is not essential here: everything can be translated into **L** with some non-lexical assumptions, as e.g. $\pi_i \Rightarrow \pi$, $s_i \Rightarrow s$.

Formally, *a categorial grammar based on a logic* $\mathcal{L}$ can be defined as a triple $G = (\Sigma, I, \alpha_0)$ such that Σ is a finite lexicon (alphabet), I is a map which assigns a finite set of types (i.e. formulas of $\mathcal{L}$) to each $v \in \Sigma$, and α_0 is a designated type. One refers to Σ, I and α_0 as the lexicon, the type lexicon and the principal type of G. In examples we write $v : \alpha$ for $\alpha \in I(v)$, and similarly for compound strings assigned type α (see below).

Usually, $\mathcal{L}$ is given in the form of a sequent system (of intuitionistic form). *A sequent* is an expression of the form $\alpha_1, \ldots, \alpha_n \Rightarrow \alpha$, where α_i, α are formulas of $\mathcal{L}$. For nonassociative logics, like **NL**, the antecedent of a sequent is a bracketed string of types (precisely: an element of the free groupoid generated by the set of formulas). Sequent systems for **L** and **NL** were proposed by Lambek [31, 32]. The axioms are (id) $\alpha \Rightarrow \alpha$ and the rules are shown in Table 1[4]. In models, each comma in the antecedent of a sequent is interpreted as product.

One says that G *assigns* type α to a string $v_1 \ldots v_n$, where all v_i belong to Σ, if there exist types $\alpha_i \in I(v_i)$, $i = 1, \ldots, n$, such that $\alpha_1, \ldots, \alpha_n \Rightarrow \alpha$ is provable in $\mathcal{L}$ (for nonassociative logics: under some bracketing of the antecedent). *The language of* G is defined as the set of all $u \in \Sigma^+$ such that G assigns α_0 to u. Often one takes an atomic type for α_0, e.g. s - the type of sentence.

[4] Clearly α, β, γ can be replaced by φ, ψ, χ.

rule	**L**	**NL**
$(\cdot\Rightarrow)$	$\frac{\Gamma,\alpha,\beta,\Gamma'\Rightarrow\gamma}{\Gamma,\alpha\cdot\beta,\Gamma'\Rightarrow\gamma}$	$\frac{\Gamma[(\alpha,\beta)]\Rightarrow\gamma}{\Gamma[\alpha\cdot\beta]\Rightarrow\gamma}$
$(\Rightarrow\cdot)$	$\frac{\Gamma\Rightarrow\alpha\quad\Delta\Rightarrow\beta}{\Gamma,\Delta\Rightarrow\alpha\cdot\beta}$	$\frac{\Gamma\Rightarrow\alpha\quad\Delta\Rightarrow\beta}{(\Gamma,\Delta)\Rightarrow\alpha\cdot\beta}$
$(\backslash\Rightarrow)$	$\frac{\Gamma,\beta,\Gamma'\Rightarrow\gamma\quad\Delta\Rightarrow\alpha}{\Gamma,\Delta,\alpha\backslash\beta,\Gamma'\Rightarrow\gamma}$	$\frac{\Gamma[\beta]\Rightarrow\gamma\quad\Delta\Rightarrow\alpha}{\Gamma[(\Delta,\alpha\backslash\beta)]\Rightarrow\gamma}$
$(\Rightarrow\backslash)$	$\frac{\alpha,\Gamma\Rightarrow\beta}{\Gamma\Rightarrow\alpha\backslash\beta}$	$\frac{(\alpha,\Gamma)\Rightarrow\beta}{\Gamma\Rightarrow\alpha\backslash\beta}$
$(/\Rightarrow)$	$\frac{\Gamma,\alpha,\Gamma'\Rightarrow\gamma\quad\Delta\Rightarrow\beta}{\Gamma,\alpha/\beta,\Delta,\Gamma'\Rightarrow\gamma}$	$\frac{\Gamma[\alpha]\Rightarrow\gamma\quad\Delta\Rightarrow\beta}{\Gamma[(\alpha/\beta,\Delta)]\Rightarrow\gamma}$
$(\Rightarrow/)$	$\frac{\Gamma,\beta\Rightarrow\alpha}{\Gamma\Rightarrow\alpha/\beta}$	$\frac{(\Gamma,\beta)\Rightarrow\alpha}{\Gamma\Rightarrow\alpha/\beta}$
(cut)	$\frac{\Gamma,\alpha,\Gamma'\Rightarrow\beta\quad\Delta\Rightarrow\alpha}{\Gamma,\Delta,\Gamma'\Rightarrow\beta}$	$\frac{\Gamma[\alpha]\Rightarrow\beta\quad\Delta\Rightarrow\alpha}{\Gamma[\Delta]\Rightarrow\beta}$

Table 1 Inference rules of sequent systems for **L** and **NL**

In rules for **NL**, $\Gamma[\Delta]$ is the result of substitution of Δ for x in the context $\Gamma[x]$. The context $\Gamma[x]$ can be defined as a bracketed string of formulas, containing one special variable x (a place for substitution). Since in **L** the antecedents of sequents are nonempty (Lambek's restriction), then Γ must be nonempty in $(\Rightarrow\backslash)$, $(\Rightarrow/)$ for **L**. Systems for logics with 1 do not admit Lambek's restriction.

Lambek [31, 32] proved *the cut-elimination theorem* for both systems: every provable sequent can be proved without (cut). This immediately yields the decidability of **NL** and **L**, since in all remaining rules the premises consist of subformulas of formulas appearing in the conclusion and the size of every premise is less than the size of the conclusion. Therefore the proof-search procedure for a cut-free proof is finite. These results remain true for several richer logics, discussed above, e.g. **MANL**, **MAL** and their versions with 1, 0 and (D). Table 2 shows axioms and rules for $\vee$, $\wedge$. A cut-free system for **DMAL1** with 0 can be found in [26].

rule	**MAL**	**MANL**
$(\vee\Rightarrow)$	$\frac{\Gamma,\alpha,\Gamma'\Rightarrow\gamma\quad\Gamma,\beta,\Gamma'\Rightarrow\gamma}{\Gamma,\alpha\vee\beta,\Gamma'\Rightarrow\gamma}$	$\frac{\Gamma[\alpha]\Rightarrow\gamma\quad\Gamma[\beta]\Rightarrow\gamma}{\Gamma[\alpha\vee\beta]\Rightarrow\gamma}$
$(\Rightarrow\vee)$	$\frac{\Gamma\Rightarrow\alpha_i}{\Gamma\Rightarrow\alpha_1\vee\alpha_2}$	same
$(\wedge\Rightarrow)$	$\frac{\Gamma,\alpha_i,\Gamma'\Rightarrow\beta}{\Gamma,\alpha_1\wedge\alpha_2,\Gamma'\Rightarrow\beta}$	$\frac{\Gamma[\alpha_i]\Rightarrow\beta}{\Gamma[\alpha_1\wedge\alpha_2]\Rightarrow\beta}$
$(\Rightarrow\wedge)$	$\frac{\Gamma\Rightarrow\alpha\quad\Gamma\Rightarrow\beta}{\Gamma\Rightarrow\alpha\wedge\beta}$	same

Table 2 Rules for $\vee$, $\wedge$ in **MAL** and **MANL**

Another consequence of the cut-elimination theorem is *the subformula property*: every provable sequent has a proof using only subformulas of formulas appearing in this sequent. Therefore **MAL** is a conservative extension of its language restricted fragments, e.g. **L**, **L** with $\wedge$ etc., and similarly for other logics, admitting cut elimination.

Unfortunately, no cut-free sequent systems are known for **NL-CL** and **L-CL**, mainly considered in this paper. Therefore the sequent systems, presented above, are not much important in what follows except for one application, mentioned below. With (cut), some sequent systems for **NL-CL** and **L-CL** can be formed easily: simply add (D), (a$\bot$), (a$\top$), ($\neg$.1) and ($\neg$.2) as new axioms to the sequent systems for **MANL** and **MAL**, respectively.

Kanazawa [23] studies categorial grammars based on **MAL**. He provides examples of types with $\wedge, \vee$, illustrating feature decomposition of types. For instance, 'walks': $(np \wedge sg)\backslash s$ and 'walk': $(np \wedge pl)\backslash s$, where np is a type of noun phrase, whereas sg and pl are types of singular and plural phrase, respectively.

In [13] types with $\vee$ are used to eliminate Lambek's non-lexical assumptions, mentioned above. Instead of them one can define $\pi = \pi_1 \vee \pi_2 \vee \pi_3$ with the same effect.

Kanazawa [23] proves that the languages generated by categorial grammars based on **MAL** are closed under finite intersections and unions. The proof essentially uses a cut-free system for **MAL**. In fact, if L_1, L_2 are ($\epsilon-$free) context-free languages, then $L_1 \cap L_2$ can be generated by a categorial grammar based on the $(\backslash, /, \wedge)-$fragment of **MAL**, i.e. the product-free **L** with $\wedge$; we denote it here by $\mathbf{L}_0$. This follows from the cut-elimination theorem and the reversibility of $(\rightarrow \wedge)$ and a restricted reversibility of $(\wedge \Rightarrow)$. Precisely: (1) if $\Gamma \Rightarrow \alpha \wedge \beta$ is provable, then both $\Gamma \Rightarrow \alpha$ and $\Gamma \Rightarrow \beta$ are provable, (2) If $\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma$ is provable in $\mathbf{L}_0$, γ does not contain $\wedge$, and $\wedge$ does not occur in the scope of $\backslash, /$, then $\Gamma, \alpha, \Delta \Rightarrow \gamma$ or $\Gamma, \beta, \Delta \Rightarrow \gamma$ is provable in $\mathbf{L}_0$. Both claims can easily be proved by induction on cut-free proofs. Let G_1, G_2 be grammars such that $L(G_1) = L_1$ and $L(G_2) = L_2$; both grammars are based on the product-free **L** and have the same lexicon Σ but no common atomic type. Let G be the grammar which to any $v \in \Sigma$ assigns the conjunction of all types assigned by G_1 and G_2 to v; its principal type equals $\alpha_1 \wedge \alpha_2$, where α_i is the principal type of G_i. It is easy to verify that $L(G) = L(G_1) \cap L(G_2)$. Therefore the generative power of categorial grammars based on **MAL** (even $\mathbf{L}_0$) is greater than those based on **L**.

$\mathbf{L}_0$ is complete with respect to language models [8], hence w.r.t. boolean residuated semigroups (see Section 2), and the latter holds for **L-CL** as well. Consequently, **L-CL** is a conservative extension of $\mathbf{L}_0$. So every grammar based on $\mathbf{L}_0$ can also be treated as a grammar based on **L-CL**. Therefore the grammars based on **L-CL** can generate some languages which are not context-free, e.g. $\{a^n b^n c^n : n \geq 1\}$.

Our grammars do not assign types to the empty string ϵ. Kuznetsov [30] shows that all context-free languages, also containing ϵ, are generated by categorial grammars based on **L1**. The construction, however, of a categorial grammar for a given language employs quite complicated types. In practice, it is easier to extend the type lexicon by assigning the principal type to ϵ, if one wants to have ϵ in the language.

In this paper we often consider the consequence relations for the given logics, i.e. provability from a set of assumptions. Categorial grammars, studied in the literature, are usually based on pure logics. This agrees with *the principle of lexicality*: all information on the particular language is contained in the type lexicon. In practice, however, this logical purity may be inconvenient. Besides non-lexical assumptions on atomic types, like those used by Lambek [33], one can approximate a stronger but less efficient logic, e.g. **L**, by a weaker but more efficient logic, e.g. **NL**, by adding to the latter some particular arrows, provable in the former only, as assumptions.

To keep this paper in a reasonable size, we cannot discuss categorial grammars in more detail. The reader is referred to [35, 36, 37, 38].

2 Algebras and frames

The algebraic models of **L** (resp. **NL**), i.e. residuated semigroups (resp. residuated groupoids), have been defined in Section 1.

By *a boolean residuated groupoid* (we write: b.r. groupoid) we mean an algebra $(A, \odot, \backslash, /, \vee, \wedge, ^-, \bot, \top)$ such that $(A, \vee, \wedge, ^-, \bot, \top)$ is a boolean algebra and $(A, \odot, \backslash, /, \leq)$ is a residuated groupoid, where $\leq$ is the boolean ordering: $a \leq b \Leftrightarrow a \vee b = b$. If $\odot$ is associative, the algebra is called *a boolean residuated semigroup* (we write: b.r. semigroup).

Remark 2 In the literature on substructural logics, one considers residuated lattices $(A, \odot, \backslash, /, 1, \vee, \wedge)$, where $(A, \vee, \wedge)$ is a lattice and $(A, \odot, \backslash, /, 1, \leq)$ is a residuated monoid, i.e. a residuated semigroup with the unit element for product ($\leq$ is the lattice ordering). They are algebraic models of **FL**. Following this terminology, one might use the term 'residuated boolean algebra' for our 'b.r. semigroup' [20, 34]. Our term, however, seems more precise and will be used in this paper. We do not assume that the algebra is unital, i.e. admits 1. If so, it is referred to as *a b.r. monoid* and for the nonassociative case *a b.r. unital groupoid*.

Like residuated lattices, b.r. groupoids and semigroups can be axiomatized by a finite set of equations, hence these classes are (algebraic) varieties. The axioms are the standard axioms for boolean algebras, i.e. the associative, commutative and distributive laws for $\vee, \wedge$, $x \vee \bot = x$, $x \wedge \top = x$, $x \vee x^- = \top$, $x \wedge x^- = \bot$ and the following axioms for $\odot, \backslash, /$:

(R1) $x \odot (x\backslash y) \leq y$, $(x/y) \odot y \leq x$
(R2) $x \leq y\backslash(y \odot x)$, $x \leq (x \odot y)/y$
(R3) $x \odot y \leq (x \vee z) \odot y$, $x \odot y \leq x \odot (y \vee z)$
(R4) $x\backslash y \leq x\backslash(y \vee z)$, $x/y \leq (x \vee z)/y$
(R5) $(x \odot y) \odot z = x \odot (y \odot z)$ (for b.r. semigroups).

Indeed, (R1)-(R4) are valid in b.r. groupoids and (R5) in b.r. semigroups. (R1), (R2) hold by (RES). Using (RES), one easily proves the monotonicity condition:

(MON) if $x \leq y$, then $z \odot x \leq z \odot y$, $x \odot z \leq y \odot z$, $z\backslash x \leq z\backslash y$, and $x/z \leq y/z$,

which yields (R3), (R4). Conversely, (RES) follow from (R1)-(R4) in lattices. First, (MON) follows from (R3), (R4). We show: $x \odot y \leq z$ iff $y \leq x\backslash z$. Assume $x \odot y \leq z$. Then $y \leq x\backslash(x \odot y) \leq x\backslash z$. Assume $y \leq x\backslash z$. Then $x \odot y \leq x \odot (x\backslash z) \leq z$.

In residuated groupoids $\backslash, /$ are antitone in the bottom argument.

(AON) if $x \leq y$, then $y\backslash z \leq x\backslash z$ and $z/y \leq z/x$

Assume $x \leq y$. Then $x \odot (y\backslash z) \leq y \odot (y\backslash z) \leq z$. This yields $y\backslash z \leq x\backslash z$, by (RES). For / the argument is dual.

In b.r. groupoids (even in l.o.r. groupoids), (MON) can be strengthened to the the following distributive laws:

$$(x \vee y) \odot z = (x \odot z) \vee (y \odot z) \quad z \odot (x \vee y) = (z \odot x) \vee (z \odot y) \tag{1}$$

$$z\backslash(x \wedge y) = (z\backslash x) \wedge (z\backslash y) \quad (x \wedge y)/z = (x/z) \wedge (y/z) \tag{2}$$

and (AON) to:

$$(x \vee y)\backslash z = (x\backslash z) \wedge (y\backslash z) \quad z/(x \vee y) = (z/x) \wedge (z/y) \tag{3}$$

Formulas of **NL-CL** and **L-CL** are formed out of propositional variables $p, q, r, \ldots$ and constants $\bot, \top$ by means of the connectives $\odot, \backslash, /, \vee, \wedge, \neg$. Given an algebra $\mathbf{A} = (A, \ldots)$, where the operations and designated elements in $\ldots$ correspond to the basic connectives and constants (in particular $^{-}$ to $\neg$), *a valuation* in $\mathbf{A}$ is a homomorphism of the (free) algebra of formulas into $\mathbf{A}$. A formula φ is said to be: (1) *true* in $\mathbf{A}$ for valuation μ, if $\mu(\varphi) = 1$, (2) *valid* in $\mathbf{A}$, if it is true in $\mathbf{A}$ for all valuations, (3) *valid in the class of algebras* $\mathcal{A}$, if it is valid in every algebra from $\mathcal{A}$. Let Φ be a set of formulas. We say that Φ *entails* φ in $\mathcal{A}$, if φ is true in every $\mathbf{A} \in \mathcal{A}$ for any μ such that all formulas in Φ are true in $\mathbf{A}$ for μ.

The connectives $\rightarrow, \leftrightarrow$ are defined as usually (in classical logic)[5].

$$\varphi \rightarrow \psi = \neg\varphi \vee \psi \quad \varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

An arrow $\varphi \Rightarrow \psi$ is said to be *true* in $\mathbf{A}$ for μ, if $\mu(\varphi) \leq \mu(\psi)$. This is equivalent to $\mu(\varphi \rightarrow \psi) = 1$. So arrows are identified with classical conditionals. Therefore the notions, defined in the preceding paragraph, can also be applied to (sets of) arrows.

A logic $\mathcal{L}$ is *weakly complete* with respect to C, if the theorems of $\mathcal{L}$ are precisely the formulas (arrows) valid in C. $\mathcal{L}$ is *strongly complete* with respect to C, if for any set of formulas (arrows) Φ and any formula (arrow) φ, φ is provable in $\mathcal{L}$ from Φ if and only if Φ entails φ in C. (φ is provable in $\mathcal{L}$ from Φ means that φ is provable in $\mathcal{L}$ enriched with all formulas (arrows) from Φ as assumptions. In opposition to axioms, assumptions need not be closed under substitutions.) Clearly weak completeness follows from strong completeness.

Theorem 1 NL-CL *(resp.* **L-CL***) is strongly complete with respect to b.r. groupoids (resp. b.r. semigroups).*

Proof The proof is routine. For soundness, one observes that all axioms are valid and all rules preserve the truth for μ in every b.r. groupoid.

For completeness, one constructs a Lindenbaum-Tarski algebra. A syntactic work is needed. One shows that the following monotonicity rules are derivable in both systems (this means: the conclusion is provable from the premise). Here $*$ represents each of the connectives $\vee, \wedge, \odot$.

[5] In first-order logic they are written $\Rightarrow, \Leftrightarrow$.

$$\frac{\varphi \Rightarrow \psi}{\chi * \varphi \Rightarrow \chi * \psi} \quad \frac{\varphi \Rightarrow \psi}{\varphi * \chi \Rightarrow \psi * \chi} \quad \frac{\varphi \Rightarrow \psi}{\neg\psi \Rightarrow \neg\varphi}$$

$$\frac{\varphi \Rightarrow \psi}{\chi\backslash\varphi \Rightarrow \chi\backslash\psi} \quad \frac{\varphi \Rightarrow \psi}{\varphi/\chi \Rightarrow \psi/\chi} \quad \frac{\varphi \Rightarrow \psi}{\psi\backslash\chi \Rightarrow \varphi\backslash\chi} \quad \frac{\varphi \Rightarrow \psi}{\chi/\psi \Rightarrow \chi/\varphi}$$

Let Φ be a set of arrows. One defines a binary relation: $\varphi \sim_\Phi \psi$ iff $\varphi \Leftrightarrow \psi$ is provable from Φ. By the monotonicity rules, $\sim_\Phi$ is a congruence on the algebra of formulas. The quotient algebra is a b.r. groupoid (resp. semigroup) for the case of **NL-CL** (resp. **L-CL**. By $[\varphi]_\Phi$ we denote the equivalence class of $\sim_\Phi$ containing φ. For the valuation μ defined by $\mu(p) = [p]_\Phi$, one gets $\mu(\varphi) = [\varphi]_\Phi$ for any formula φ. One proves: $[\varphi]_\Phi \leq [\psi]_\Phi$ iff $\varphi \Rightarrow \psi$ is provable from Φ.

Consequently, $\varphi \Rightarrow \psi$ is provable from Φ if and only if $[\varphi \to \psi]_\Phi = [\top]_\Phi$, which is equivalent to $\mu(\varphi \to \psi) = [\top]_\Phi$. Therefore all arrows in Φ are true for μ in the quotient algebra. If $\varphi \Rightarrow \psi$ is not provable from Φ, then $[\varphi \to \psi] \neq [\top]_\Phi$, which means that $\varphi \Rightarrow \psi$ is not true for μ; so Φ does not entail $\varphi \to \psi$ in b.r. groupoids (semigroups). □

Remark 3 A formula φ is said to be *provable* in the system, if $\top \Rightarrow \varphi$ is provable. For either system, there holds: $\varphi \Rightarrow \psi$ is provable from Φ if and only if $\varphi \to \psi$ is so. Indeed, from $\varphi \Rightarrow \psi$ we get $\neg\varphi \vee \varphi \Rightarrow \neg\varphi \vee \psi$, by monotonicity, hence $\top \Rightarrow \varphi \to \psi$, by lattice laws, ($\neg$.2), (cut-1) and the definition of $\to$. Conversely, from $\top \Rightarrow \varphi \to \psi$ we get $\varphi \wedge \top \Rightarrow \varphi \wedge (\neg\varphi \vee \psi)$, by monotonicity and the definition of $\to$, hence $\varphi \Rightarrow \varphi \wedge \psi$, by (D), bounded lattice laws, monotonicity and ($\neg$.1), which yields $\varphi \Rightarrow \psi$, by (a$\wedge$) and (cut-1).

The proof of Theorem 1 can be adapted for several other logics. In a similar way one proves that **NL** (resp. **L**) is strongly complete w.r.t. residuated groupoids (resp. residuated semigroups), **NL1** (resp. **L1**) w.r.t. residuated unital groupoids (resp. residuated monoids), **MANL** (resp. **MAL**) w.r.t. l.o.r. groupoids (resp. semigroups) and so on.

Remark 4 Every formula of our logics can be translated into a term of the first-order theory of the corresponding algebras. By $t(\varphi)$ we denote the translation of φ. It is reasonable to change the symbols for operations $\vee, \wedge$ in terms and algebras: write $\cup$ for $\vee$ and $\cap$ for $\wedge$. For instance, $t(p \vee \neg p) = x \cup x^-$. By completeness, $\varphi \Rightarrow \psi$ is valid in the class if and only if $t(\varphi) \leq t(\psi)$ is so. By strong completeness, $\varphi \Rightarrow \psi$ is provable from Φ if and only if $t(\varphi) \leq t(\psi)$ follows from $\{t(\chi) \leq t(\chi') : (\chi \Rightarrow \chi') \in \Phi\}$ in this class. For a finite Φ, the latter condition is equivalent to the validity of the Horn formula: the conjunction of all formulas $t(\chi) \leq t(\chi')$, for $\chi \Rightarrow \chi'$ in Φ, implies $t(\varphi) \leq t(\psi)$. As a consequence, formal proofs of arrows in a system can be replaced by algebraic proofs of the corresponding first-order formulas, which are often shorter and easier. For logics with $\vee, \wedge$, atomic formulas with $\leq$ can be replaced by equations $s = t$, hence Horn formulas by quasi-equations: $s_1 = t_1 \wedge \cdots \wedge s_n = t_n \Rightarrow s = t$.

We consider some special constructions of b.r. semigroups (groupoids). Given a groupoid $(G, \cdot)$, one defines operations $\odot, \backslash, /$ on $\mathcal{P}(G)$.

$$X \odot Y = \{a \cdot b : a \in X, b \in Y\}$$

$$X\backslash Y = \{b \in G : X \odot \{b\} \subseteq Y\} \quad X/Y = \{a \in G : \{a\} \odot Y \subseteq X\}$$

$(\mathcal{P}(G), \odot, \backslash, /, \subseteq)$ is a residuated groupoid. Clearly $\mathcal{P}(G)$ is a boolean algebra of sets, hence this construction yields a b.r. groupoid, which we refer to as *the powerset algebra* of $(G, \cdot)$. If $(G, \cdot)$ is a semigroup, then this construction yields a b.r. semigroup. If $(G, \cdot, 1)$ is a unital groupoid (resp. monoid), then this construction yields a b.r. unital groupoid (resp. b.r. monoid), where $\{1\}$ is the unit element for $\odot$.

One also considers relation algebras $\mathcal{P}(W^2)$ with $\odot, \backslash, /$ defined as follows.

$$R \odot S = \{(x, y) \in W^2 : \exists_z ((x, z) \in R \wedge (z, y) \in S)\}$$

$$R\backslash S = \{(z, y) \in W^2 : R \odot \{(z, y)\} \subseteq S\} \quad R/S = \{(x, z) \in W^2 : \{(x, z)\} \odot S \subseteq R\}$$

They are algebraic models of **L1-CL**; $\mathrm{Id}_W = \{(x, x) : x \in W\}$ is the unit element for $\odot$. Clearly $\odot$ is the relative product (often written as $R \circ S$ or $R; S$). For **L-CL**, one considers algebras $\mathcal{P}(U)$, where $U \subseteq W^2$ is a transitive relation (then $\mathcal{P}(U)$ is closed under $\odot$; in definitions of $\backslash, /$ one replaces W^2 with U). The strong completeness of **L** w.r.t. the latter algebras was shown in [3].

In linguistics, the intended models of **L** are powerset algebras of $(\Sigma^+, \cdot)$, where Σ^+ is the set of all nonempty finite strings over a (finite) alphabet Σ and $\cdot$ is the concatenation of strings (this operation is associative). Subsets of Σ^+ are called $\epsilon-$free languages on Σ. One often refers to these algebras as *language models*. For **NL**, strings are replaced by bracketed strings (phrase structures). For bracketed strings x, y one defines: $x \cdot y = (x, y)$ (in examples, comma can be omitted). For instance, $a \cdot (b, c) = (a, (b, c))$, but we write (Jane (meets John)). By $\Sigma^{(+)}$ we denote the set of all bracketed strings on Σ. More precisely, $(\Sigma^{(+)}, \cdot)$ is the free groupoid generated by Σ, and $(\Sigma^+, \cdot)$ is the free semigroup generated by Σ. By ϵ we denote the empty string. $\Sigma^* = \Sigma^+ \cup \{\epsilon\}$ is the set of all finite strimgs on Σ. Σ^* with concatenation and ϵ is the free monoid generated by Σ. We also define $\Sigma^{(*)} = \Sigma^{(+)} \cup \{\epsilon\}$ and assume $\epsilon \cdot x = x = x \cdot \epsilon$ for $x \in \Sigma^{(*)}$; this yields the free unital groupoid generated by Σ. The powerset algebras of Σ^* and $\Sigma^{(*)}$ are the intended models of **L1** and **NL1**, respectively; $\{\epsilon\}$ is the unit element.

By Theorem 1, **L-CL** and **NL-CL** are (strongly) sound with respect to powerset algebras of semigroups and groupoids, respectively, and consequently to language models. **L** (resp. **NL**) is strongly complete with respect to powerset algebras of semigroups [8] (resp. groupoids [25]). The proofs of the latter results employ some labeled deductive systems.

As a consequence, **L-CL** (resp. **NL-CL**) is a *strongly conservative* extension of **L** (resp. **NL**). This means: for any set of arrows Φ and an arrow $\varphi \Rightarrow \psi$ in the language of the weaker system, this arrow is provable from this set in the weaker system if and only if it is so in the stronger system (for $\Phi = \emptyset$, this defines a *conservative* extension). We prove it for **L**. The 'only-if' part is obvious. For the 'if' part, assume that $\phi \Rightarrow \psi$ is not provable from Φ in **L**. Since **L** is strongly complete with respect to powerset algebras of semigroups, there exist a semigroup $(G, \cdot)$ and a valuation

μ in $\mathcal{P}(G)$ such that all arrows in Φ are true for μ but $\varphi \Rightarrow \psi$ is not. The powerset algebra can be expanded to a b.r. semigroup. So Φ does not entail $\varphi \Rightarrow \psi$ in b.r. semigroups. By Theorem 1, $\varphi \Rightarrow \psi$ is not provable from Φ in **L-CL**. This argument also works for **NL** versus **NL-CL**. The very result has already been proved in [21]; the proof uses frame models (see below).

Pentus [40] proves the weak completeness of **L** with respect to language models $\mathcal{P}(\Sigma^+)$ (the proof is quite involved). This does not hold for **NL** and language models $\mathcal{P}(\Sigma^{(+)})$. We recall an example of Došen [15]. The arrow $((p \odot q)/r) \odot r \Rightarrow p \odot r$ is valid in these algebras. For assume $a \in \mu(((p \odot q)/r) \odot r)$. Then $a = b \cdot c$, where $b \in \mu((p \odot q)/r)$ and $c \in \mu(r)$. Hence $b \cdot c \in \mu(p \odot q)$, which yields $b \in \mu(p)$, since in $\Sigma^{(+)}$ b, c are the only elements such that $a = b \cdot c$. Consequently $a \in \mu(p \odot r)$. This arrow, however, is not provable in **NL**, since it is not valid in residuated groupoids. Take the free group generated by $\{p, q, r\}$. Every group is a residuated semigroup (hence groupoid), with $a \odot b = a \cdot b$ (write ab), $a\backslash b = a^{-1}b$, $a/b = ab^{-1}$, and $\leq$ being the identity relation. For μ defined by $\mu(p) = p$ and similarly for q, r, one gets $\mu(((p \odot q)/r) \odot r) = pqr^{-1}r \neq pr = \mu(p \odot r)$.

The same example shows that **NL-CL** is not weakly complete with respect to language models $\mathcal{P}(\Sigma^{(+)})$: this arrow is not provable in **NL-CL**, since **NL-CL** is a conservative extension of **NL**. The following, stronger proposition implies that **L-CL** is not weakly complete with respect to language models $\mathcal{P}(\Sigma^+)$ and **NL1-CL** (resp. **L1-CL**) is not weakly complete with respect to language models $\mathcal{P}(\Sigma^*)$ (resp. $\mathcal{P}(\Sigma^{(*)})$).

Proposition 1 **NL-CL** *(resp.* **L-CL***) is not weakly complete with respect to powerset algebras of groupoids (resp. semigroups). Also* **NL1-CL** *(resp.* **L1-CL***) is not weakly complete with respect to powerset algebras of unital groupoids (resp. monoids).*

Proof Let $(G, \cdot)$ be a groupoid. In the powerset algebra, $\emptyset/X = \emptyset$ for any nonempty $X \subseteq G$. We consider the arrow $\bot/(p\backslash(p \odot p)) \Rightarrow \bot$. Since $p \Rightarrow p\backslash(p \odot p)$ is provable in **NL**, then $\mu(p\backslash(p \odot p)) \neq \emptyset$ for any valuation μ in $\mathcal{P}(G)$. This holds, if $\mu(p) \neq \emptyset$; if $\mu(p) = \emptyset$, then $\mu(p\backslash(p \odot p)) = G \neq \emptyset$. Therefore our arrow is valid in powerset algebras of groupoids. This arrow, however, is not provable in **L1-CL** (the strongest logic), since it is not valid in relation algebras $\mathcal{P}(W^2)$. Indeed, let $W = \{a, b, c\}$ (three different elements), $\mu(p) = \{(a, b), (b, c)\}$. Then $\mu(p \odot p) = \{(a, c)\}$, $\mu(p\backslash(p \odot p)) = \{(b, c)\} \cup \{(a, x); x \in W\}$ and $\mu(\bot/(p\backslash(p \odot p))) = \{(x, c) : x \in W\} \neq \emptyset$. □

This proof also shows that **NL** with constants $\bot, \top$ and axioms (a$\bot$), (a$\top$) is not weakly complete with respect to powerset algebras of groupoids, and similarly for **L**, **NL1** and **L1**, extended in this way.

Example 1 Here is another arrow valid in powerset algebras of groupoids but not provable in **L1-CL**: $(\neg q)/(p\backslash(p \odot p)) \Rightarrow \neg(q/(p\backslash(p \odot p)))$. By (2) and the law: $x \leq y^-$ iff $x \wedge y = \bot$, the arrow is valid in powerset algebras of groupoids (resp. provable in **L1-CL**) if and only if $\bot/(p\backslash(p \odot p)) \Rightarrow \bot$ is so.

Example 2 Došen's example does not work for **NL1**; $((p \odot q)/r) \odot r \Rightarrow p \odot r$ is not valid in language models $\mathcal{P}(\Sigma^{(*)})$. Indeed, for $\mu(r) = \{\epsilon\}$, if this arrow is true, then $\mu(p \odot q) \subseteq \mu(p)$, which need not be true. A good example is $q \odot (1/(p\backslash p)) \Rightarrow (1/(p\backslash p)) \odot q$. This arrow is valid in powerset algebras of unital groupoids. Indeed $1 \Rightarrow p\backslash p$ is provable in **NL1**, hence $1/(p\backslash p) \Rightarrow 1$ is so. In these models, $\mu(1/(p\backslash p))$ equals $\emptyset$ or $\{1\}$, hence it commutes with $\mu(q)$. This arrow, however, is not valid in relation algebras $\mathcal{P}(W^2)$. Define W, $\mu(p)$ as in the proof of Proposition 1 and $\mu(q) = \{(a, b)\}$. Then $\mu(p\backslash p) = \{(b, b), (c, c)\} \cup \{(a, x) : x \in W\}$, $\mu(1/(p\backslash p)) = \{(b, b), (c, c)\}$. So for μ the left-hand side of the arrow equals $\{(a, b)\}$ and the right-hand side is empty. Therefore **NL1** is not weakly complete w.r.t. powerset algebras of unital groupoids, hence w.r.t. language models $\mathcal{P}(\Sigma^{(*)})$. Since relation algebras $\mathcal{P}(W^2)$ are models of **L1**, this example also shows that **L1** is not weakly complete w.r.t. powerset algebras of monoids, hence w.r.t. language models $\mathcal{P}(\Sigma^*)$.

We turn to frame models, characteristic of Kripke semantics for modal logics. A pair (W, R) such that $R \subseteq W^3$ is called *a (ternary) relational frame*. On $\mathcal{P}(W)$ one defines operations $\odot, \backslash, /$ (sometimes they are written $\odot_R, \backslash_R, /_R$).

$$X \odot Y = \{u \in W : R(u, v, w) \text{ for some } v \in X, w \in Y\}$$

$$X\backslash Y = \{w \in W : X \odot \{w\} \subseteq Y\} \quad X/Y = \{v \in W : \{v\} \odot Y \subseteq X\}$$

Accordingly: $w \in X\backslash Y$ iff for all $u, v \in W$, if $R(u, v, w)$ and $v \in X$ then $u \in Y$. Also: $v \in X/Y$ iff for all $u, w \in W$, if $R(u, v, w)$ and $w \in Y$ then $v \in X$. We have: $X \odot Y \subseteq Z$ iff $Y \subseteq X\backslash Z$ iff $X \subseteq Z/Y$. Indeed, each condition is equivalent in first-order logic to the formula: $\forall_{u,v,w}(R(u, v, w) \wedge v \in X \wedge w \in Y \Rightarrow u \in Z)$. Consequently, $(\mathcal{P}(W), \odot, \backslash, /, \cup, \cap, ^-, \emptyset, W)$ is a b.r. groupoid. One refers to this algebra as *the complex algebra* of the frame (W, R).

A frame (W, R) is said to be *associative*, if $(X \odot Y) \odot Z = X \odot (Y \odot Z)$ for all $X, Y, Z \subseteq W$. If a frame is associative, then its complex algebra is a b.r. semigroup. The associativity of (W, R) is equivalent to the following condition.

$$\forall_{u,x,y,z \in W}[\exists_v(R(v, x, y) \wedge R(u, v, z)) \Leftrightarrow \exists_w(R(u, x, w) \wedge R(w, y, z))] \qquad (4)$$

The next proposition is equivalent to the strong completeness of Hilbert-style systems for **NL-CL** and **L-CL** with respect to frame models, proved in [21]. We, however, outline a different proof, using Theorem 1 and the following representation theorem for b.r. groupoids (semigroups).

Theorem 2 *([6]) Every b.r. groupoid (resp. b.r. semigroup) is isomorphic to a subalgebra of the complex algebra of some (resp. associative) frame.*

Proposition 2 **NL-CL** *(resp.* **L-CL***) is strongly complete with respect to the complex algebras of (resp. associative) relational frames.*

Proof We prove the strong completeness of **NL-CL**. If $\varphi \Rightarrow \psi$ is provable from Φ in this logic, then Φ entails $\varphi \Rightarrow \psi$ in b.r. groupoids, hence in the complex algebras of frames. Assume that $\varphi \Rightarrow \psi$ is not provable from Φ. By Theorem 1, there exist a b.r. groupoid **A** and a valuation μ in **A** such that all arrows in Φ are true for μ in **A**, but $\varphi \Rightarrow \psi$ is not. By the representation theorem, there exists a frame (W, R)

and a monomorphism h from **A** to the complex algebra of (W, R). Clearly $h \circ \mu$ is a valuation in the latter algebra. All arrows in Φ are true for $h \circ \mu$, but $\varphi \Rightarrow \psi$ is not. Therefore Φ does not entail $\varphi \Rightarrow \psi$ in the complex algebras of frames. A similar argument works for **L-CL**. □

We recall some main steps of the proof of the representation theorem, since we need them later. Let **A** be a b.r. groupoid.. One defines the canonical frame. W consists of all ultrafilters on the boolean algebra underlying **A**. For $F, G, H \in W$ one defines: $R(F, G, H)$ iff $G \odot H \subseteq F$. Here $X \odot Y = \{a \odot b : a \in X, b \in Y\}$ for $X, Y \subseteq A$. The canonical embedding $h : A \mapsto \mathcal{P}(W)$ is defined by $h(a) = \{F \in W : a \in F\}$. One shows that h is a monomorphism of **A** into the complex algebra of (W, R).

Crucial Lemma: Let F_1, F_2 be proper filters and let F be an ultrafilter in the boolean algebra such that $F_1 \odot F_2 \subseteq F$. Then, there exist ultrafilters G_1, G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \odot G_2 \subseteq F$.

We prove $h(a \odot b) = h(a) \odot_R h(b)$. We show $\subseteq$. Let $F \in h(a \odot b)$. Then $a \odot b \in F$. Since F is an ultrafilter, then $a \odot b \neq \bot$, hence $a \neq \bot$ and $b \neq \bot$. One defines $F_a = \{x \in A : a \leq x\}$ and F_b similarly. F_a, F_b are proper filters and $F_a \odot F_b \subseteq F$. There exist ultrafilters G_1, G_2 as in the lemma. We have $G_1 \in h(a)$, $G_2 \in h(b)$ and $R(F, G_1, G_2)$. This yields $F \in h(a) \odot_R h(b)$. We show $\supseteq$. Let $F \in h(a) \odot_R h(b)$. Then, $R(F, G_1, G_2)$, i.e. $G_1 \odot G_2 \subseteq F$ for some $G_1 \in h(a)$, $G_2 \in h(b)$. Since $a \odot b \in G_1 \odot G_2$, then $a \odot b \in F$, and consequently $F \in h(a \odot b)$. For other steps the reader is referred to [6].

If **A** is a b.r. semigroup, then $\odot$ in the complex algebra of the canonical frame is associative. We only prove ($\Rightarrow$) of (4). Let $F_1, F_2, F_3, H \in W$. Assume that $R(G_1, F_1, F_2)$ and $R(H, G_1, F_3)$, i.e. $F_1 \odot F_2 \subseteq G_1$ and $G_1 \odot F_3 \subseteq H$, for some $G_1 \in W$. Since the powerset operation $\odot$ on $\mathcal{P}(A)$ is associative and preserves $\subseteq$, then $F_1 \odot F_2 \odot F_3 \subseteq H$. Define $F = \{x \in A : \exists_{y,z}(y \in F_2 \wedge z \in F_3 \wedge y \odot z \leq x)\}$. Clearly F is a proper filter, $F_2 \odot F_3 \subseteq F$, and $F_1 \odot F \subseteq H$. By the lemma, there exists $G_2 \in W$ such that $F \subseteq G_2$ and $F_1 \odot G_2 \subseteq H$. This yields $R(G_2, F_2, F_3)$ and $R(H, F_1, G_2)$. This finishes the proof.

Remark 5 **DMANL** and **DMAL** (also: with $\bot, \top$) are strongly complete w.r.t. the complex algebras of relational frames and associative frames, respectively. This can be proved like Proposition 2, using the following representation theorem [6]: every (also: bounded) d.l.o.r. groupoid (resp. semigroup) is isomorphic to a subalgebra of the complex algebra of some (resp. associative) relational frame. (Precisely, we mean the $^-$−free reduct of the complex algebra.) The proof is similar to that of Theorem 2 except that ultrafilters are replaced with prime filters. As a consequence, **NL-CL** (resp. **L-CL**) is a strongly conservative extension of **DMANL** (resp. **DMAL**), also with bounds. Interestingly, this does not hold for logics with 1.

Proposition 3 NL1-CL *(resp.* **L1-CL***) is a non-conservative extension of* **DMANL1** *(resp.* **DMAL1***), also with bounds.*

Proof In b.r. unital groupoids, if $a \leq 1$, then $a \odot a = a$. We show it. Assume $a \leq 1$. Define $b = a^- \wedge 1$. We have $1 = \top \wedge 1 = (a \vee a^-) \wedge 1 = (a \wedge 1) \vee b = a \vee b$. For

$x, y \leq 1$, $x \odot y \leq x \wedge y$ (indeed, $x \odot y \leq x \odot 1 = x$ and $x \odot y \leq 1 \odot y = y$). So $a \odot b = \bot$. This yields $a = a \odot 1 = a \odot (a \vee b) = (a \odot a) \vee \bot = a \odot a$.

It follows that $x \wedge 1 \leq (x \wedge 1) \odot (x \wedge 1)$ is valid in b.r. unital groupoids, and the corresponding arrow is provable in **NL1-CL**. This arrow, however, is not provable in **DMAL1** (even with bounds). It suffices to observe that $x \wedge 1 \leq (x \wedge 1) \odot (x \wedge 1)$ is not valid in bounded d.l.o.r. monoids, e.g. in MV-algebras[6], i.e. algebras of many-valued logics of Łukasiewicz. Consider the closed interval $[0, 1] \subseteq \mathbb{R}$ (the stamdard model of $Ł_\infty$), where $x \wedge y = \min(x, y)$, $x \odot y = \max(0, x + y - 1)$; the number 1 is both $\top$ and the multiplicative unit. Then $x \wedge 1 = x$, but $x \leq x \odot x$ is not true for $x = \frac{1}{2}$. □

In fact, in b.r. unital groupoids $x \odot y = x \wedge y$ for all $x, y \leq 1$. As above, $x \odot y \leq x \wedge y$. Also $x \wedge y = (x \wedge y) \odot (x \wedge y) \leq x \odot y$.

At the end of this section, we consider some operations definable in b.r. groupoids. First, one defines the De Morgan dual of $\odot$ and its dual residual operations.

$$a \bullet b = (a^- \odot b^-)^- \quad a \backslash^\bullet b = (a^- \backslash b^-)^- \quad a /^\bullet b = (a^- / b^-)^- \tag{5}$$

There hold dual residuation laws.

(RES$\bullet$) For all elements a, b, c, $c \leq a \bullet b$ iff $a \backslash^\bullet c \leq b$ iff $c /^\bullet b \leq a$.

In the complex algebra of (W, R):

$$X \bullet Y = \{u \in W : \forall_{v,w}(R(u, v, w) \Rightarrow v \in X \vee w \in Y)\}$$

$$X \backslash^\bullet Y = \{w \in W : \exists_{u,v}(R(u, v, w) \wedge v \notin X \wedge u \in Y)\}$$

$$X /^\bullet Y = \{v \in W : \exists_{u,w}(R(u, v, w) \wedge w \notin Y \wedge u \in X)\}$$

Our definition of $\odot$ in the complex algebra of (W, R) takes the first element of the triple (u, v, w) as an element of $X \odot Y$. The reason is to make this definition compatible with the standard definition of $\Diamond X$ in analogous algebras for modal logics. Some authors, however, prefer the third element of (u, v, w) in this role. One can define $\odot_2$ and $\odot_3$ as follows.

$$X \odot_2 Y = \{v \in W : \exists_{w,u}(R(u, v, w) \wedge w \in X \wedge u \in Y)\}$$

$$X \odot_3 Y = \{w \in W : \exists_{u.v}(R(u, v, w) \wedge u \in X \wedge v \in Y)\}$$

We also set $\odot_1 = \odot$. By $\backslash_i, /_i$ we denote the residual operations for $\odot_i$ and by $\bullet_i, \backslash^\bullet_i, /^\bullet_i$ the corresponding dual operations. As observed in several papers, e.g. [27], the new operations are definable in terms of $\odot, \backslash, /$ and $^-$.

$$X \odot_2 Y = (Y^- / X)^- \quad X \backslash_2 Y = (Y^- \odot X)^- \quad X /_2 Y = X^- \backslash Y^-$$

$$X \odot_3 Y = (Y \backslash X^-)^- \quad X \backslash_3 Y = X^- / Y^- \quad X /_3 Y = (Y \odot X^-)^-$$

The following equivalences:

[6] Every MV-algebra is a bounded d.l.o.r. commutative monoid, where $1 = \top$, $0 = \bot$.

$$X \cap (Y \odot Z) = \emptyset \text{ iff } Y \cap (Z \odot_2 X) = \emptyset \text{ iff } Z \cap (X \odot_3 Y) = \emptyset$$

show that $\odot_2$ is the left and $\odot_3$ the right conjugate of $\odot$ in the sense of Jónsson and Tarski; see [20].

Sedlár and Tedder [43] study **DMANL** enriched with $\odot_2, \odot_3$ and their residuals; they provide complete (w.r.t. frames) axiom systems for some language restricted fragments, leaving the problem for the full logic open. By Proposition 2, **NL-CL** is a conservative extension of all complete logics of this kind.

3 H-systems and modal logics

In this section we present Hilbert-style systems (H-systems) for **NL-CL**, **L-CL** and their extensions. In these systems one derives provable formulas; see Remark 3. We treat them as classical modal logics with binary modalities $\odot, \backslash, /$. We aslo consider their extensions by new axioms, natural in the frameworks of modal and substructural logics. At the end, we show how the standard method of filtration can be adjusted for binary modalities.

3.1 Unary modalities

First, we recall a H-system for $\mathbf{K}_t$ (the minimal tense logic [7]), just to illuminate a close relationship between **NL-CL** and $\mathbf{K}_t$.

$\mathbf{K}_t$ is a classical modal logic with unary modalities $\Box, \Box^{\downarrow}$. Dual modalities $\Diamond, \Diamond^{\downarrow}$ are defined as follows: $\Diamond\varphi = \neg\Box\neg\varphi$, $\Diamond^{\downarrow}\varphi = \neg\Box^{\downarrow}\neg\varphi$. In tense logics, one usually writes F, P, G, H for $\Diamond, \Diamond^{\downarrow}, \Box, \Box^{\downarrow}$, respectively.

The corresponding frames are of the form (W, R), where $R \subseteq W^2$. *A frame model* is a triple $M = (W, R, V)$ such that (W, R) is a frame and V is a map from the set of propositional variables to $\mathcal{P}(W)$. The truth predicate $u \models_M \varphi$, where $u \in W$ and φ is a formula, is defined as usually.

($\models p$) $u \models_M p$ iff $u \in V(p)$
($\models \neg$) $u \models_M \neg\varphi$ iff $u \not\models_M \varphi$
($\models \wedge$) $u \models_M \varphi \wedge \psi$ iff $u \models_M \varphi$ and $u \models_M \psi$
($\models \Box$) $u \models_M \Box\varphi$ iff $v \models_M \varphi$ for any $v \in W$ such that $R(u, v)$
($\models \Box^{\downarrow}$) $u \models_M \Box^{\downarrow}\varphi$ iff $v \models_M \varphi$ for any $v \in W$ such that $R(v, u)$

φ is *valid* in M, if $w \models_M \varphi$ for all $w \in W$, and in the frame (W, R), if it is valid in all models (W, R, V).

$\mathbf{K}_t$ can be presented as the following H-system [7]. The axioms are all tautologies of classical logic (in the modal language) and the modal axioms[7]:

[7] Precisely, in [7] $\Diamond$ is primitive and $\Box$ is defined. The additional axiom $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ is needed.

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
($K^{\downarrow}$) $\Box^{\downarrow}(\varphi \to \psi) \to (\Box^{\downarrow}\varphi \to \Box^{\downarrow}\psi)$
($A\Diamond\Box^{\downarrow}$) $\Diamond\Box^{\downarrow}\varphi \to \varphi$
($A\Box^{\downarrow}\Diamond$) $\varphi \to \Box^{\downarrow}\Diamond\varphi$

Its inference rules are *modus ponens* and two *necessitation rules*.

$$\text{(MP)}\ \frac{\varphi \to \psi \quad \varphi}{\psi} \quad \text{(RN)}\ \frac{\varphi}{\Box\varphi} \quad (\text{RN}^{\downarrow})\ \frac{\varphi}{\Box^{\downarrow}\varphi}\,.$$

One derives the monotonicity rules for modalities.

$$\text{(r-MON1)}\ \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \quad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$$

$$\text{(r-MON2)}\ \frac{\varphi \to \psi}{\Box^{\downarrow}\varphi \to \Box^{\downarrow}\psi} \quad \frac{\varphi \to \psi}{\Diamond^{\downarrow}\varphi \to \Diamond^{\downarrow}\psi}$$

The following *residuation rule* is derivable:

$$\text{(r-RES1)}\ \frac{\Diamond\varphi \to \psi}{\overline{\varphi \to \Box^{\downarrow}\psi}}$$

The top-down part of (r-RES1) is derived by (r-MON2), ($A\Box^{\downarrow}\Diamond$) and the bottom-up part by (r-MON1)), ($A\Diamond\Box^{\downarrow}$). If $\Diamond$ is admitted as primitive, then (r-RES1) can replace the four modal axioms, written above, and (RN), ($RN^{\downarrow}$); see the next subsection, where an analogous claim is proved for logics with binary modalities. (With $\Box$ primitive, the additional axiom $\Box\varphi \Leftrightarrow \neg\Diamond\neg\varphi$ is needed.)

The modal axiom scheme:

(B) $\varphi \to \Box\Diamond\varphi$

is valid[8] in (W, R) if and only if R is symmetrical: $R(u, v)$ implies $R(v, u)$. In models M, based on symmetric frames, $u \models_M \Box\varphi$ iff $u \models_M \Box^{\downarrow}\varphi$. We prove a syntactic counterpart of this equivalence.

Proposition 4 *In* $\mathbf{K}_t$ *(B) (as a scheme) is deductively equivalent to* $\Box\varphi \leftrightarrow \Box^{\downarrow}\varphi$.

Proof The second scheme yields (B), by ($A\Box^{\downarrow}\Diamond$). For the converse, (B) yields $\Diamond\Box\varphi \to \varphi$, hence $\Box\varphi \to \Box^{\downarrow}\varphi$, by (r-RES1). Also, $\Box^{\downarrow}\varphi \to \Box\Diamond\Box^{\downarrow}\varphi$ is an instance of (B), and $\Box\Diamond\Box^{\downarrow}\varphi \to \Box\varphi$, by ($A\Diamond\Box^{\downarrow}$) and (r-MON1). This yields $\Box^{\downarrow}\varphi \to \Box\varphi$. □

Consequently, in $\mathbf{K}_t$ with (B) (and its extensions) $\Box$ and $\Box^{\downarrow}$ collapse, and one can remove $\Box^{\downarrow}$ from the language. One omits all axioms and rules for $\Box^{\downarrow}$.

[8] This means that all instances of (B) are valid.

3.2 Binary modalities

Now we turn to binary modalities. They are precisely $\odot, \backslash, /$ of **L**, added to the standard language of classical propositional logic. So one can define other modalities, e.g. $\bullet, \backslash^\bullet, /^\bullet, \bullet_i$ etc. as in Section 2. A (ternary) frame has been defined there. A model is a triple $M = (W, R, V)$ such that (W, R) is a frame and V maps the set of variables into $\mathcal{P}(W)$. The truth definition is standard for variables and classical connectives. One defines:

($\models \odot$) $u \models_M \varphi \odot \psi$ iff for some $v, w \in W$, $R(u, v, w)$, $v \models_M \varphi$ and $w \models_M \psi$
($\models \backslash$) $w \models_M \varphi\backslash\psi$ iff for all $u, v \in W$, if $R(u, v, w)$ and $v \models_M \varphi$ then $u \models_M \psi$
($\models /$) $v \models_M \varphi/\psi$ iff for all $u, w \in W$, if $R(u, v, w)$ and $w \models_M \psi$ then $u \models_M \varphi$

The notions of validity in a model and in the frame are defined as in Subsection 3.1. We also define entailment in models on a class of ternary frames $\mathcal{F}$. A set of formulas Φ *entails* a formula φ *in models on* $\mathcal{F}$, if φ is valid in every model $M = (W, R, V)$ such that $(W, R) \in \mathcal{F}$ and all formulas from Φ are valid in M. We write $\Phi \models_{\mathcal{F}} \varphi$ for this entailment relation[9], and similarly $\Phi \models_{\mathcal{A}} \varphi$ for the entailment relation in a class of algebras $\mathcal{A}$ (see Section 2).

For a model $M = (W, R, V)$, one defines $\mu_M(\varphi) = \{u \in W : u \models_M \varphi\}$. It is easy to show that μ_M is a valuation in the complex algebra of (W, R); furthermore, every valuation in this complex algebra equals μ_M for some model $M = (W, R, V)$. Clearly φ is valid in M if and only if $\mu_M(\varphi) = W$ ($W = \top$ in this algebra). Let $\mathcal{F}$, C, and $\mathcal{A}$ denote now the class of ternary frames, the class of their complex algebras, and the class of b.r. groupoids, respectively.. The following equivalences are true for any Φ and φ.

$$\Phi \models_{\mathcal{F}} \varphi \text{ iff } \Phi \models_C \varphi \text{ iff } \Phi \models_{\mathcal{A}} \varphi \tag{6}$$

The second equivalence follows from Theorem 2. (6) also hold for the associative case: $\mathcal{F}$ is the class of associative frames, C of their complex algebras, and $\mathcal{A}$ of b.r. semigroups.

Since in b.r. groupoids $\odot$ distributes over $\vee$ and satisfies $\bot \odot a = \bot = a \odot \bot$, then it can be treated as a binary normal possibility operator. In a ternary frame, R can be interpreted as an accessibility relation in the following sense: $R(u, v, w)$ means that from the world (state) u one can access a pair of worlds (states) (v, w) in one step.

There are many natural examples of ternary frames. Every groupoid $(G, \cdot)$ determines the frame (G, R), where: $R(u, v, w)$ iff $u = v \cdot w$, for $u, v, w \in G$. The complex algebra of this frame coincides with the powerset algebra $\mathcal{P}(G)$. Given a set W with a partial function f from W^2 to W (the domain of f is contained in W^2), one obtains the frame (W, R_f), where: $R_f(u, v, w)$ iff $f(v, w)$ is defined and equals u. Every relation algebra $\mathcal{P}(W^2)$ coincides with the complex algebra of the frame (W^2, R_f), where f is the composition of pairs: $f((x, y), (z, u))$ is defined iff $y = z$; $f((x, y), (y, u)) = (x, u)$.

Example 3 Another example employs formal logics. We consider a propositional logic $\mathcal{L}$ whose all rules have two premises. Its formulas will be denoted by α, β, γ.

[9] This relation should not be confused with the stronger relation: Φ entails φ in $\mathcal{F}$, if φ is valid in every frame from $\mathcal{F}$ such that all formulas from Φ are valid in this frame.

We define a frame (W, R) such that W is the set of formulas of this logic and R is defined as follows: $R(\alpha, \beta, \gamma)$ iff α can be derived from β and γ by one application of some rule. Then, modal formulas of **NL-CL** can encode proof schemes in $\mathcal{L}$. Let $\mathcal{L}$ be the logic of positive implication with (MP) as the only rule and the following axioms.

(A1) $\alpha \to (\beta \to \alpha)$

(A2) $[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$

We write a proof of $\alpha \to \alpha$.

1	$[\alpha \to ((\beta \to \alpha) \to \alpha)] \to [(\alpha \to (\beta \to \alpha)) \to (\alpha \to \alpha)]$	(A2)
2	$\alpha \to ((\beta \to \alpha) \to \alpha)$	(A1)
3	$(\alpha \to (\beta \to \alpha)) \to (\alpha \to \alpha)$	MP 1,2
4	$\alpha \to (\beta \to \alpha)$	(A1)
5	$\alpha \to \alpha$	MP 3,4

We consider a model (W, R, V), where W, R are as above and $V(p)$ (resp. $V(q)$) is the set of all axioms (A1) (resp. (A2)). Then, $\alpha \to \alpha \models_M (q \odot p) \odot p$ and every formula γ such that $\gamma \models_M (q \odot p) \odot p$ is of this form for some α. If, additionally, $V(r)$ is the set of provable formulas, then $r \leftrightarrow p \vee q \vee (r \odot r)$ is valid in M.

To obtain a more precise description of proof schemes of $\mathcal{L}$ with rules $r_1, \ldots, r_n$, the relation R from Example 3 could be replaced by relations $R_1, \ldots, R_n$, each corresponding to one rule. The resulting modal logic admits several products $\odot_1, \ldots, \odot_n$ and the related residual operators. If one-premise rules r_j appeared in $\mathcal{L}$, it would be reasonable to represent them by binary relations $R_j \subseteq W^2$ (as in Subsection 3.1), corresponding to unary operators $\Diamond_j$. Such multi-modal logics can be useful in applications. We, however, discuss logics with one product. Most results can easily be generalized for the case of many products and unary modalities, at least if no special connections between them are assumed.

Now we discuss H-systems. The system **PNL** of Kaminski and Francez [21] is formulated in the language of classical propositional logic enriched by $\odot, \backslash, /$. Its axioms are all tautologies of classical logic (in the extended language). Its rules are (MP) and (r1), (r2) from Section 1 with $\Rightarrow$ replaced by $\to$. **PL** also admits the associative law for $\odot$:

$$(\varphi \odot \psi) \odot \chi \leftrightarrow \varphi \odot (\psi \odot \chi) \tag{7}$$

as an axiom. These systems are strongly complete w.r.t. the corresponding classes of ternary frames.

Theorem 3 *[21] For any set of formulas Φ and any formula φ, φ is provable from Φ in* **PNL** *(resp.* **PL***) if and only if Φ entails φ in models on (resp. associative) ternary frames.*

Proof The present proof is different from that in [21]. Like in the proof of Theorem 1, one shows that **PNL** (resp. **PL**) is strongly complete w.r.t. b.r. groupoids (resp. b.r. semigroups): φ is provable from Φ in the system if and only if $\Phi \models_{\mathcal{A}} \varphi$, where $\mathcal{A}$ is the class of b.r. groupoids (resp. b.r. semigroups). Then, one applies (6). ◻

By Proposition 2, **PNL** (resp. **PL**) is simply a H-system for **NL-CL** (resp. **L-CL**). Both systems yield the same provable formulas (see Remark 3) and the same consequence relation, i.e. provability from assumptions. Clearly $\bot$ and $\top$ can be defined in these H-systems: $\bot = p \wedge \neg p$, $\top = p \vee \neg p$, for some fixed p. We use the latter acronyms in what follows.

Another H-system for **NL-CL** is similar to $\mathbf{K}_t$. It is convenient to take $\bullet$ instead of $\odot$ as a primitive modal operator. Clearly $\odot$ becomes definable: $\varphi \odot \psi = \neg(\neg\varphi \bullet \neg\psi)$. Rules (r1), (r2) can be replaced by the following axioms and rules.

(K1) $(\varphi \to \psi) \bullet \chi \to (\varphi \bullet \chi \to \psi \bullet \chi)$
(K2) $\varphi \bullet (\psi \to \chi) \to (\varphi \bullet \psi \to \varphi \bullet \chi)$
(K\) $\varphi\backslash(\psi \to \chi) \to (\varphi\backslash\psi \to \varphi\backslash\psi)$
(K/) $(\psi \to \chi)/\varphi \to (\psi/\varphi \to \chi/\varphi)$
(A1\) $\varphi \odot (\varphi\backslash\psi) \to \psi$
(A1/) $(\varphi/\psi) \odot \psi \to \varphi$
(A2\) $\psi \to \varphi\backslash(\varphi \odot \psi)$
(A2/) $\varphi \to (\varphi \odot \psi)/\psi$

$$\text{(RN1)}\ \frac{\varphi}{\psi \bullet \varphi} \quad \text{(RN2)}\ \frac{\varphi}{\varphi \bullet \psi}$$

$$\text{(RN}\backslash\text{)}\ \frac{\varphi}{\psi\backslash\varphi} \quad \text{(RN/)}\ \frac{\varphi}{\varphi/\psi}$$

There is a clear analogy between (K) and (K1), (K2), between (K$^{\downarrow}$) and (K\), (K/), between (A$\Diamond\Box^{\downarrow}$) and (A1\), (A1/), between (A$\Box^{\downarrow}\Diamond$) and (A2\), (A2/), between (RN) and (RN1), (RN2), and between (RN$^{\downarrow}$) and (RN\), (RN/).

The following monotonicity rules are easily derivable in both axiomatizations.

(MON$\bullet$) from $\varphi \to \psi$ infer $\chi \bullet \varphi \to \chi \bullet \psi$ and $\varphi \bullet \chi \to \psi \bullet \chi$
(MON$\odot$) from $\varphi \to \psi$ infer $\chi \odot \varphi \to \chi \odot \psi$ and $\varphi \odot \chi \to \psi \odot \chi$
(MON\) from $\varphi \to \psi$ infer $\chi\backslash\varphi \to \chi\backslash\psi$ and $\psi\backslash\chi \to \varphi\backslash\chi$
(MON/) from $\varphi \to \psi$ infer $\varphi/\chi \to \psi/\chi$ and $\chi/\psi \to \chi/\varphi$

Both H-systems for **NL-CL** are equivalent (the provability relation is the same). We outline a proof. S_1 stands for **PNL** from [21] and S_2 for the system similar to $\mathbf{K}_t$.

First, (r1), (r2) are derivable in S_2. We derive (r1). Assume $\varphi \odot \psi \to \chi$. By (RN\), we get $\varphi\backslash(\varphi \odot \psi \to \chi)$, hence $\varphi\backslash(\varphi \odot \psi) \to \varphi\backslash\chi$, by (K\) and (MP). This yields $\psi \to \varphi\backslash\chi$, by (A2\) and classical logic. Assume $\psi \to \varphi\backslash\chi$. We get $\varphi \odot \psi \to \varphi \odot (\varphi\backslash\chi)$, by (MON$\odot$), which yields $\varphi \odot \psi \to \chi$, by (A1\) and classical logic.

Second, axioms (K1)-(A2/) are provable and rules (RN1)-(RN/) are derivable in S_1. It is easy to prove (A1\), (A1/), (A2\) and (A2/). Using (r1), (r2), one proves the distributive law.

$$\varphi \odot (\psi \vee \chi) \leftrightarrow (\varphi \odot \psi) \vee (\varphi \odot \chi) \quad (\psi \vee \chi) \odot \varphi \leftrightarrow (\psi \odot \varphi) \vee (\chi \odot \varphi) \tag{8}$$

In S_1 $\bullet$ is defined: $\varphi \bullet \psi = \neg(\neg\varphi \odot \neg\psi)$. Using (8), monotonicity rules and classical logic, one proves the following law.

$$\varphi \bullet (\psi \wedge \chi) \leftrightarrow (\varphi \bullet \psi) \wedge (\varphi \bullet \chi) \quad (\psi \wedge \chi) \bullet \varphi \leftrightarrow (\psi \bullet \varphi) \wedge (\chi \bullet \varphi) \tag{9}$$

From $(\varphi \to \psi) \wedge \varphi \to \psi$ one obtains (K1), using (MON$\bullet$), (9) and classical logic. (K2) is obtained similarly. (K$\backslash$) and (K/) are obtained in a similar way, using the same classical tautology, (MON$\backslash$), (MON/) and the following laws analogous to (2), easily provable in S_1.

$$\varphi\backslash(\psi \wedge \chi) \leftrightarrow (\varphi\backslash\psi) \wedge (\varphi\backslash\chi) \quad (\psi \wedge \chi)/\varphi \leftrightarrow (\psi/\varphi) \wedge (\chi/\varphi) \tag{10}$$

We derive (RN1). From $(\neg\psi) \odot \bot \leftrightarrow \bot$ one obtains $\psi \bullet \top \leftrightarrow \top$. Assume φ. Then $\top \to \varphi$ by classical logic. Hence $\psi \bullet \top \to \psi \bullet \varphi$ by (MON$\bullet$), which yields $\top \to \psi \bullet \varphi$ by classical logic, and consequently, $\psi \bullet \varphi$ by (MP). The derivation of (RN2) is similar. We derive (RN$\backslash$). Assume φ. Then $\psi \odot \top \to \varphi$ by classical logic. Hence $\top \to \psi\backslash\varphi$ by (r1), which yields $\psi\backslash\varphi$ by (MP). The derivation of (RN/) is similar.

S_2 enriched by the associative law for $\bullet$ is a H-system for **L-CL**. Clearly this law for $\bullet$ implies this law for $\odot$, and conversely.

3.3 Other modal axioms

At first we consider some analogues of the symmetry axiom (B). For $R \subseteq W^3$, the symmetry property of a binary relation has different counterparts. We list three.

(WS) for all $u, v, w \in W$, if $R(u, v, w)$ then $R(u, w, v)$ (weak symmetry)
(Cy) for all $u, v, w \in W$, if $R(u, v, w)$ then $R(w, u, v)$ (cyclicity)
(FS) for all $u_1, u_2, u_3 \in W$, if $R(u_1, u_2, u_3)$ then $R(u_{i_1}, u_{i_2}, u_{i_3})$ for any permutation (i_1, i_2, i_3) of $(1, 2, 3)$ (full symmetry)

Clearly (FS) is equivalent to the conjunction of (WS) and (Cy). (WS) corresponds to the commutative law.

$$\text{(COM)}\ \varphi \odot \psi \leftrightarrow \psi \odot \varphi$$

Precisely, (COM) is valid in the frame (W, R) if and only if R satisfies (WS). Like in algebras, the scheme (COM) is deductively equivalent to the scheme $\varphi\backslash\psi \leftrightarrow \psi/\varphi$. Modal logics admitting (COM) are said to be *commutative*. In commutative logics $\backslash$, / collapse in one operator, which we denote by $\to_\Box$ (similar to $\Box^\downarrow$), just to distinguish it from $\to$. One omits all axioms and rules for / and writes $\to_\Box$ for $\backslash$ in the remaining ones. Like Theorem 1 and Theorem 3, one proves that Commutative **NL-CL** (resp. **L-CL**) is strongly complete w.r.t. commutative b.r. groupoids (resp. semigroups) and models on (resp. associative) ternary frames, satisfying (WS).

Returning to Example 3, let us note that the relation R, defined there, satisfies (WS), if the premises of a rule are treated as a set (their order is inessential).

The frames satisfying (Cy) are said to be *cyclic*. We look at the corresponding logics closer, since they can be regarded as classical counterparts of cyclic linear logics. In particular, Cyclic **MALL** of Yetter [48] can be presented as **MAL1** with 0, $\bot$ (see Section 1), admitting the cyclic axiom: $\varphi\backslash 0 \Leftrightarrow 0/\varphi$. So two substructural negations collapse in one, written $^\sim$. Cyclic **MALL** also assumes the double negation law:

$\varphi^{\sim\sim} \Leftrightarrow \varphi$; the contraposition rule: *from* $\varphi \Rightarrow \psi$ *infer* $\psi^{\sim} \Rightarrow \varphi^{\sim}$ is derivable. So the resulting negation is a De Morgan negation. One obtains the following contraposition law[10]: (CONT) $\varphi^{\sim}/\psi \Leftrightarrow \varphi\backslash\psi^{\sim}$.

$$\varphi^{\sim}/\psi \Leftrightarrow (\varphi\backslash 0)/\psi \Leftrightarrow \varphi\backslash(0/\psi) \Leftrightarrow \varphi\backslash\psi^{\sim}$$

The second $\Leftrightarrow$ follows from the associativity of $\odot$. One can consider weaker logics of this kind, e.g. without constants 1, 0 (also in algebras) and/or with nonassociative product. In them, $^{\sim}$ is a primitive connective; the double negation law and (CONT) are admitted as axioms and the contraposition rule is assumed. In this way, from **MANL** one obtains the nonassociative version of Cyclic **MALL** without multiplicative constants [19, 11]. Let us refer to this logic as Cyclic **MANL**.

Remark 6 In the literature on linear logics and Lambek calculi, the extensions admitting the double negation law are often referred to as 'classical', like in [19, 11]. This usage of 'classical' seems misleading: this law holds in genuine nonclassical logics, e.g. many-valued logics and relevance logics. As in the literature on substructural logics, the term 'cyclic' is preferred here.

By Cyclic **NL-CL** we mean **NL-CL** enriched with the following axiom scheme.

$$\text{(CONT}\neg\text{)}\ \neg\varphi/\psi \leftrightarrow \varphi\backslash\neg\psi$$

In its arrow version $\leftrightarrow$ is replaced by $\Leftrightarrow$. Clearly (CONT¬) is valid in (W, R) if and only if in the complex algebra of (W, R), for all $X, Y \subseteq W$ $X \odot_2 Y = X \odot_3 Y$; the latter condition is equivalent to (Cy) for R. Cyclic **NL-CL** is an extension of Cyclic **MANL**, if one translates the latter's $\varphi^{\sim}$ as $\neg\varphi$. Therefore classical negation behaves in the former like cyclic negation in linear logics. In particular, with $\backslash, /$ it fulfils contraposition laws. We prove the strong completeness of Cyclic **NL-CL** w.r.t. cyclic ternary frames.

Proposition 5 *For any set of formulas* Φ *and any formula* φ*,* φ *is provable from* Φ *in Cyclic* **NL-CL** *if and only if* Φ *entails* φ *in models on cyclic ternary frames.*

Proof A b.r. groupoid **A** is said to be *cyclic*, if $a^{-}/b = a\backslash b^{-}$ for all $a, b \in A$. Like Theorem 1, one proves that Cyclic **NL-CL** is strongly complete w.r.t. cyclic b.r. groupoids. Since (CONT¬) is valid in cyclic ternary frames, then the complex algebras of these frames are cyclic b.r. groupoids. Consequently, Cyclic **NL-CL** is sound w.r.t. the complex algebras of cyclic frames.

For completeness, like Theorem 2 one shows that every cyclic b.r. groupoid is isomorphic to a subalgebra of the complex algebra of some cyclic frame. It suffices to observe that, if **A** is a cyclic b.r. groupoid, then the canonical frame (W, R) is cyclic. Indeed, assume $R(F, G, H)$, i.e. $G \odot H \subseteq F$. We show $R(H, F, G)$, i.e. $F \odot G \subseteq H$. Suppose $F \odot G \nsubseteq H$. There exists $a \in F$, $b \in G$ such that $a \odot b \notin H$. Then $(a \odot b)^{-} \in H$. We have $(a \odot b)^{-} = b\backslash a^{-}$. Indeed, $c \leq b\backslash a^{-}$ iff $b \odot c \leq a^{-}$ iff

[10] (CONT) is equivalent to other laws of this kind, e.g. $\varphi\backslash\psi \Leftrightarrow \varphi^{\sim}/\psi^{\sim}$.

$b \leq a^-/c$ iff $b \leq a\backslash c^-$ iff $a \odot b \leq c^-$ iff $c \leq (a \odot b)^-$, for any $c \in A$. Therefore $b \odot (a \odot b)^- \leq a^-$, by (R1). Since $b \odot (a \odot b)^- \in F$ by the assumption, we get $a^- \in F$, which contradicts $a \in F$. □

One defines $a \odot_2 b = (b^-/a)^-$ and $a \odot_3 b = (b\backslash a^-)^-$ in any b.r. groupoid. Then $a \odot b = a \odot_2 b = a \odot_3 b$ is valid in cyclic b.r. groupoids. The proof of Proposition 5 implicitly uses $a \odot b = a \odot_3 b$. Also $a \bullet b = b/a^- = b^-\backslash a$ is valid these algebras. The corresponding logical equivalences are analogous to the scheme $\Box\varphi \leftrightarrow \Box^{\downarrow}\varphi$ for unary modalities. In linear logics, one defines the operation *par* (a De Morgan dual of $\odot$). Its classical counterpart, here denoted by $\bullet'$, satisfies in algebras $a \bullet' b = b \bullet a$. One obtains $a \bullet' b = a/b^- = a^-\backslash b$, which yields $a/b = a \bullet' b^-$, $a\backslash b = a^- \bullet' b$ (definitions of /, \ in terms of par and negation in algebras of cyclic linear logics).

Analogues of Proposition 5 can be proved for Cyclic **L-CL**, i.e. **L-CL** with (CONT¬), and versions with multiplicative constants and (COM). Cyclic **L1-CL** is an extension of Cyclic **MALL**. A closer examination of these logics and their applications must be deferred to another paper. In cyclic commutative logics, corresponding to frames satisfying (FS), (CONT¬) takes the following form.

$$(\varphi \to_{\Box} \neg\psi) \leftrightarrow (\psi \to_{\Box} \neg\varphi)$$

For logics with 1, the corresponding frames are of the form (W, R, E), where (W, R) is as above and $E \subseteq W$ satisfies:

$$\exists_{e \in E} R(u, e, w) \Leftrightarrow u = w \quad \exists_{e \in E} R(u, v, e) \Leftrightarrow u = v$$

for all $u, v, w \in W$. This yields $E \odot X = X = X \odot E$, for any $X \subseteq W$, in the complex algebra of (W, R). Accordingly, this algebra is a b.r. unital groupoid. All results of this section hold for the logics, discussed here, enriched with 1, but we omit all details.

The modal axiom scheme (T) $\varphi \to \Diamond\varphi$ is analogous to the following scheme of contraction laws in substructural logics.

$$\text{(CON)}\ \varphi \to \varphi \odot \varphi$$

(T) is valid in (W, R), $R \subseteq W^2$, if and only if R is reflexive. Similarly, (COM) is valid in (W, R), $R \subseteq W^3$, if and only if $R(u, u, u)$ holds for any $u \in W$; we say that this frame is reflexive. Like Theorem 3 one proves the strong completeness of **NL-CL** (resp. **L-CL**) w.r.t. models on (resp. associative) reflexive ternary frames. The algebraic condition corresponding to (CON) is: $a \leq a \odot a$ for any element a (one says that $\odot$ is *square-increasing*). In the proof the following observation is essential: if $\odot$ in **A** is square increasing, then in the canonical frame $R(F, F, F)$ holds for any ultrafilter F. We show it. Assume that $\odot$ in **A** is self-increasing. Then $a \wedge b \leq (a \wedge b) \odot (a \wedge b) \leq a \odot b$. Hence, for all $a, b \in F$, $a \odot b \in F$, which yields $F \odot F \subseteq F$, i.e. $R(F, F, F)$.

Let us note that the stronger schemes $\varphi \to \varphi \odot \psi$,$\psi \to \varphi \odot \psi$ lead to the inconsistent logic. Fix a provable formula φ_0. In the first scheme replace φ with φ_0

and ψ with $\varphi_0\backslash\psi$; then use (A1\). This yields $\varphi_0 \to \psi$, for any ψ, and consequently, every formula ψ is provable.

The converse schemes:

$$\text{(LWE)}\ \varphi \odot \psi \to \varphi \quad \varphi \odot \psi \to \psi$$

express the algebraic conditions: $a \odot b \leq a$, $a \odot b \leq b$, for all elements a, b (one says that $\odot$ is *decreasing*). These conditions correspond to the left-weakening rules in sequent systems for substructural logics.

Their analogue for $\Diamond$ is $\Diamond\varphi \to \varphi$. This scheme is valid in (W, R), $R \subseteq W^2$, if and only if, for any $u, v \in W$, $R(u, v)$ implies $u = v$, i.e. $R \subseteq \mathrm{Id}_W$. The resulting logic is not interesting as a modal logic: one proves $\Diamond\varphi \leftrightarrow \varphi \wedge \Diamond\top$. The situation is similar for (LWE). These schemes are valid in (W, R), $R \subseteq W^3$, if and only if, for any $u, v, w \in W$, $R(u, v, w)$ implies $u = v = w$. For such models M, one obtains the following truth condition.

$$u \models_M \varphi \odot \psi \text{ iff } u \in U, u \models_M \varphi \text{ and } u \models_M \psi, \text{ where } U = \{u : R(u, u, u)\}$$

Therefore the following scheme is valid.

$$\varphi \odot \psi \leftrightarrow \varphi \wedge \psi \wedge \top \odot \top \tag{11}$$

In fact, in **NL-CL** (11) is deductively equivalent to (LWE). We prove the algebraic version of this equivalence.

Proposition 6 *For any b.r. groupoid* **A***, the following conditions are equivalent: (i)* $\odot$ *is decreasing, (ii)* $a \odot b = a \wedge b \wedge \top \odot \top$ *for all* $a, b \in A$.

Proof Clearly (i) follows from (ii). We prove the converse. Assume (i). Then $a \odot b \leq a \wedge b$. We obtain:

$$a \wedge b \wedge \top \odot \top = a \wedge b \wedge (a \vee a^-) \odot (b \vee b^-) =$$
$$(a \wedge b \wedge a \odot b) \vee (a \wedge b \wedge a \odot b^-) \vee (a \wedge b \wedge a^- \odot b) \vee (a \wedge b \wedge a^- \odot b^-) =$$
$$a \odot b \vee \bot \vee \bot \vee \bot = a \odot b$$

This yields (ii). □

The resulting logic amounts to classical logic with a new variable p_U and definitions:

$$\varphi \odot \psi = \varphi \wedge \psi \wedge p_U \quad \varphi\backslash\psi = \varphi \wedge p_U \to \psi \quad \varphi/\psi = \psi \wedge p_U \to \varphi$$

Then $\top \odot \top \leftrightarrow p_U$ is provable. In b.r. unital groupoids, $\top \odot \top = \top$; hence, if $\odot$ is decreasing, then $a \odot b = a \wedge b$ and $a\backslash b = b/a = a^- \vee b$ for all elements a, b. Accordingly **NL1-CL** with (LWE) amounts to classical logic. Clearly **L-CL** with (LWE) equals **NL-CL** with (LWE), since the latter's product is associative (and commutative).

Remark 7 Lambek calculi and linear logics are often interpreted as logics of actions (programs); see e.g. [18, 46]. For sentences of natural language, expressing an action, $\odot$ can be interpreted as the conjunction (superposition) of actions, which need not be commutative. ‘Susan met John and John bought flowers’ is not synonymous to ‘John bought flowers and Susan met John’. One might claim that the truth of either sentence implies the truth of both ‘Susan met John’ and ‘John bought flowers’. It is tempting to employ **L-CL** with the axiom-scheme $\varphi \odot \psi \to \varphi \wedge \psi$, equivalent to (LWE), for logical analysis of such sentences. The preceding paragraph, however, shows that this logic is too strong. Its product almost coincides with classical conjunction; with 1 even coincides. Therefore a weaker logic must be employed, e.g. **DMAL** or **DMAL1**, either with (LWE). Another option is **L-CL** or **L1-CL** with (LWE) replaced by the rule: *from* $\varphi \odot \psi$ *infer* $\varphi \wedge \psi$.

The modal axiom (4): $\Diamond\Diamond\varphi \to \Diamond\varphi$, valid in transitive binary frames, can be adapted for binary modalities in several ways. We leave an analysis of these options to further research.

3.4 Filtration

Kaminski and Francez [21] prove the strong finite model property of **PNL** w.r.t. models on ternary frames: if φ is not provable from finite Φ in **PNL**, then Φ does not entail φ in models on finite ternary frames. By Theorem 2, this result is equivalent to the strong finite model property of **NL-CL** w.r.t. b.r. groupoids, established in [13, 10]. Nonetheless, the direct proof in [21] is interesting: it uses a filtration of a frame model whose worlds are certain sets of formulas. This filtration, however, is defined in a nonstandard way: worlds are subsets of a finite set of formulas. Here we briefly explain how to adapt the standard method of filtration (as in [7] for unary modalities) for logics with $\odot, \backslash, /$.

Let Γ be a set of formulas of **NL-CL**. Γ is said to be *suitable*, if it is closed under subformulas and satisfies the conditions:

($\backslash\odot$) if $\varphi\backslash\psi \in \Gamma$ then $\varphi \odot (\varphi\backslash\psi) \in \Gamma$,
($/\odot$) if $\varphi/\psi \in \Gamma$ then $(\varphi/\psi) \odot \psi \in \Gamma$.

Every finite set Γ_0 can be extended to a finite suitable set Γ. First, add to Γ_0 all subformulas of formulas from Γ_0. Second, add to the obtained set new formulas, according to ($\backslash\odot$) and ($/\odot$) (these steps do not iterate). The resulting set Γ is suitable.

Let $M = (W, R, V)$ be a model, where $R \subseteq W^3$. Let Γ be a set of formulas. We define an equivalence relation $\sim_\Gamma \subseteq W^2$.

$$u \sim_\Gamma v \text{ iff for any } \varphi \in \Gamma, u \models_M \varphi \Leftrightarrow v \models_M \varphi \tag{12}$$

By $[u]_\Gamma$ we denote the equivalence class of $\sim_\Gamma$ containing u; the subscript Γ is often omitted. We define $W_\Gamma = \{[u] : u \in W\}$.

A filtration of M through Γ is defined as a model $M^f = (W_\Gamma, R^f, V^f)$ such that $V^f(p) = \{[u] : u \in V(p)\}$ and $R^f \subseteq (W_\Gamma)^3$ satisfies the following conditions for all $u, v, w \in W$:

(f1) if $R(u, v, w)$, then $R^f([u], [v], [w])$,
(f2) if $R^f([u], [v], [w])$, $\varphi \odot \psi \in \Gamma$, $v \models_M \varphi$ and $w \models_M \psi$, then $u \models_M \varphi \odot \psi$.

The lemma below explains the role of $(\backslash\odot)$ and $(/\odot)$,

Lemma 1 *Let Γ satisfy $(\backslash\odot)$ (resp. $(/\odot)$), and let $R^f \subseteq (W_\Gamma)^3$ satisfy (f2). Thus, for all φ, ψ, if $\varphi\backslash\psi \in \Gamma$ (resp. $\varphi/\psi \in \Gamma$), $R^f([u], [v], [w])$, $v \models_M \varphi$ (resp. $w \models_M \psi$), and $w \models_M \varphi\backslash\psi$ (resp. $v \models_M \varphi/\psi$), then $u \models_M \psi$ (resp. $u \models_M \varphi$).*

Proof We prove the first part only. Assume that $R^f([u], [v], [w])$, $\varphi\backslash\psi \in \Gamma$, $v \models_M \varphi$ and $w \models_M \varphi\backslash\psi$. Since $\varphi \odot (\varphi\backslash\psi) \in \Gamma$, then $u \models \varphi \odot (\varphi\backslash\psi)$, by (f2). Since (A1\) is valid in M, then $u \models_M \psi$. □

Lemma 2 *(Filtration Lemma) Let Γ be a suitable set of formulas. Let M^f be a filtration of $M = (W, R, V)$ through Γ. Then, for all $\chi \in \Gamma$ and $u \in W$, $u \models_M \chi$ if and only if $[u]_\Gamma \models_{M^f} \chi$.*

Proof Induction on $\chi \in \Gamma$. Let $\chi = p$. If $u \models_M p$, then $u \in V(p)$, which yields $[u] \in V^f(p)$, hence $[u] \models_{M^f} p$. Assume $[u] \models_{M^f} p$. Then $[u] \in V^f(p)$, hence $u \sim_\Gamma v$ for some $v \in V(p)$. Since $v \models_M p$, then $u \models_M p$.

The arguments for classical connectives are routine. We consider the cases $\chi = \varphi \odot \psi$ and $\chi = \varphi\backslash\psi$.

Assume $u \models_M \varphi \odot \psi$. There exist v, w such that $R(u, v, w)$, $v \models_M \varphi$ and $w \models_M \psi$. We have $R^f([w], [u], [v])$, by (f1), and $[v] \models_{M^f} \varphi$, $[w] \models_{M^f} \psi$, by the induction hypothesis. Consequently $[u] \models_{M^f} \varphi \odot \psi$. Assume $[u] \models_{M^f} \varphi \odot \psi$. There exist v, w such that $R^f([u], [v], [w])$, $[v] \models_{M^f} \varphi$ and $[w] \models_{M^f} \psi$. So $v \models_M \varphi$, $w \models_M \psi$, by the induction hypothesis. By (f2), $u \models_M \varphi \odot \psi$.

Assume $w \models_M \varphi\backslash\psi$. Let $R^f([u], [v], [w])$ and $[v] \models_{M^f} \varphi$. Then $v \models_M \varphi$, by the induction hypothesis, hence $u \models_M \psi$, by Lemma 1. This yields $[u] \models_{M^f} \psi$. Consequently $[w] \models_{M^f} \varphi\backslash\psi$. Assume $[w] \models_{M^f} \varphi\backslash\psi$. Let $R(u, v, w)$ and $v \models_M \varphi$. Then $[v] \models_{M^f} \varphi$, by the induction hypothesis. Since $R^f([u], [v], [w])$ by (f1), then $[u] \models_{M^f} \psi$. Consequently $u \models_M \psi$, by the induction hypothesis. We have shown $w \models_M \varphi\backslash\psi$. □

For a set Γ closed under subformulas, we define the smallest and the largest filtration of M through Γ:

(sf) $R^s([u], [v], [w])$ iff there exist $u' \in [u]$, $v' \in [v]$, $w' \in [w]$ such that $R(u', v', w')$,
(lf) $R^l([u], [v], [w])$ iff for all φ, ψ, if $\varphi \odot \psi \in \Gamma$, $v \models_M \varphi$ and $w \models_M \psi$, then $u \models_M \varphi \otimes \psi$.

It is easy to verify that R^s and R^l satisfy (f1), (f2) and, for any R^f satisfying (f1), (f2), $R^s \subseteq R^f \subseteq R^l$. Our proof of the following theorem uses filtration in the sense, defined above.

Theorem 4 *[21]* **NL-CL** *possesses the strong finite model property w.r.t. models on ternary frames.*

Proof let Φ be a finite set of formulas, and let φ be a formula not provable from Φ in **NL-CL**. Let Γ_0 be the set of all formulas appearing in $\Phi \cup \{\varphi\}$. We extend Γ_0 to a finite suitable set Γ. By Theorem 3, there exists a model $M = (W, R, V)$ such that all formulas from Φ are valid but φ is not valid in M. We construct a filtration M^f of M through Γ. One can put $R^f = R^s$ or $R^l = R^l$ (both work). By Lemma 2 all formulas from Φ are valid but φ is not valid in M^f. Since W_Γ is finite, then Φ does not entail φ in models on finite frames. □

Actually, this yields the *bounded* finite model property. By *the size* of Γ ($s(\Gamma)$) we mean the number of variables and connectives occurring in formulas from Γ (for connectives, we count their occurrences). The number of all subformulas of formulas from Γ is not greater than $s(\Gamma)$. If Γ is the smallest suitable set containing Γ_0, then Γ consists of at most $2s(\Gamma_0$ formulas. Consequently, W_Γ has at most 2^n elements, where $n = 2s(\Gamma_0)$. We obtain: φ is provable from Φ in **NL-CL** if and only if Φ entails φ in models on ternary frames with at most 2^n elements. Clearly this implies the decidability of the provability from finite sets Φ in **NL-CL**.

Analogous results can be obtained for all nonassociative extensions, discussed in Subsection 3.3. The following observations are crucial. If $R \subseteq W^2$ satisfies (WS) (resp. (Cy), (FS)), then R^s satisfies (WS) (resp. (Cy), (FS)). If R is reflexive, then R^s is reflexive. If $R \subseteq \{(u, u, u) : u \in W\}$, then $R^s \subseteq \{([u], [u], [u]) : [u] \in W_\Gamma\}$ (this is not very useful, since the corresponding logic reduces to classical logic).

They cannot be adapted, at least directly, for associative versions of these logics. **L-CL** is undecidable (see Subsection 4), hence it does not possess *the finite model property*: every unprovable formula can be falsified in a finite model. Consequently, filtration does not preserve associativity: for $R \subseteq W^3$, satisfying (4), R^f need not satisfy (4).

4 Decidability and complexity

NL-CL is decidable, and similarly for the provability from a finite set Φ [13, 21]. More is known. The provability in the pure logic **NL-CL** is PSPACE-complete. Lin and Ma [34] prove it by: (1) a polynomial translation of **K**, i.e. $\mathbf{K}_t$ without $\Box^\downarrow$, in **NL-CL**, (2) a polynomial translation of **NL-CL** in $\mathbf{K}_t$ (in two steps: first, **NL-CL** in $\mathbf{K}_{2t}$, i.e. $\mathbf{K}_t$ with the second pair of modalities $\Box_2, \Box_2^\downarrow$: axioms and rules for them copy those for $\Box, \Box^\downarrow$, second, a polynomial translation of $\mathbf{K}_{2t}$ in $\mathbf{K}_t$). Since **K** and $\mathbf{K}_t$ are PSPACE-complete [7], (1) implies that **NL-CL** is PSPACE-hard and (2) that it is PSPACE.

Let $\mathcal{A}$ be a class of algebras. By Eq($\mathcal{A}$) (resp. Queq($\mathcal{A}$)) we denote the set of all equations (resp. quasi-equations) valid in $\mathcal{A}$; we refer to the first-order language of $\mathcal{A}$ (see Section 2). *A universal sentence* is a sentence $\forall_{x_1,\ldots,x_n}\varphi$, where φ is a quantifier-free first-order formula. *The universal theory of* $\mathcal{A}$ is defined as the set of all universal sentences valid in $\mathcal{A}$ and denoted $\mathrm{Th}_U(\mathcal{A})$.

By G (resp. SG) we denote the class of groupoids (resp. semigroups), by RG (resp. RSG) the class of residuated groupoids (resp. semigroups), by DLRG the class of d.l.o.r. groupoids, and by bDLRG the class of bounded d.l.o.r. groupoids. BRG (resp. BRSG) denotes the class of b.r. groupoids (resp. semigroups).

Shkatov and van Alten [44] prove that Th_U(bDLRG) is EXPTIME-complete; this proof also yields the EXPTIME-completeness of Queq(bDLRG). The same authors [45] prove the EXPTIME-completeness of the universal theory of normal modal algebras. Details are too involved to be discussed here. It follows (see Remark 4) that the provability from finite sets in **DMANL** with $\bot, \top$ is EXPTIME-complete. Since **NL-CL** is a strongly conservative extension of **DMANL** with $\bot, \top$ (see Remark 5), then the provability from finite sets in **NL-CL** is EXPTIME-hard. It is also EXPTIME. The proof from [44] that Th_U(bDLRG) is EXPTIME, which uses some characterization of the partial algebras being subalgebras of algebras in bDLRG, can be adjusted for BRG, like it is made in [45] for normal modal algebras with unary modal operators. Therefore Qeq(BRG) is EXPTIME, which shows that the provability from finite sets in **NL-CL** is EXPTIME-complete.

For associative logics the situation radically changes. **L-CL** is undecidable. This was explicitly stated by Kurucz et al. [28] who proved a more general result. In the next paper [29], not referring to the Lambek calculus, the same authors proved a closely related result: classical propositional logic enriched wit a binary modality, distributing over disjunction, is undecidable. Since the undecidability of **L-CL** is an important result for our subject-matter, we present a proof below. Our proof essentially follows that in [29] (which simplifies the approach of [28]), but further simplifies it and repairs an error, namely a wrong definition of the equation $e(q)$, encoding a quasi-equation q.

We consider quasi-equations $s_1 = t_1 \wedge \cdots \wedge s_n = t_n \Rightarrow s_0 = t_0$, $n \geq 0$, in the first-order language of semigroups, i.e. $\odot$ is the only operation symbol[11] which can appear in terms s_i, t_i. For a class of algebras $\mathcal{A}$, admitting an operation $\odot$, $\mathrm{Queq}_\odot(\mathcal{A})$ denotes the set of all quasi-equations of this form valid in $\mathcal{A}$. Clearly $\mathrm{Queq}_\odot$(SG)=Queq(SG).

Queq(SG) is undecidable. This amounts to the classical result of computability theory: *the word problem for semigroups is undecidable*. We will show that Queq(SG) can be encoded in Eq(BRSG), which yields the undecidability of Eq(BRSG). As a consequence, **L-CL** is undecidable (see Remark 4).

Lemma 3 *Queq(SG)=*$\mathit{Queq}_\odot$*(BRSG)*

Proof $\subseteq$ is obvious. To prove $\supseteq$ we observe that every semigroup $(G, \odot)$ is isomorphic to a subalgebra of the semigroup reduct of a b.r. semigroup, namely $\mathcal{P}(G)$ with operations $\odot, \backslash, /$ defined as in Section 2 (except that $\cdot$ is replaced with $\odot$). The map $h(a) = \{a\}$ is a monomorphism of $(G, \odot)$ into $\mathcal{P}(G)$. $\square$

[11] We use $\odot$ instead of $\cdot$ for the semigroup operation, since the former is used in b.r. semigroups.

We define a term $\sigma(x)$ in the first-order language of BRSG (as it has been noted in Remark 4, in terms $\cup, \cap$ are used for $\vee, \wedge$).

$$\sigma(x) := x \cup \top \odot x \cup x \odot \top \cup \top \odot x \odot \top \tag{13}$$

Lemma 4 *Let* $\mathbf{A}$ *be a b.r. semigroup. For any* $a \in A$*: (i)* $a \leq \sigma(a)$*, (ii)* $\top \odot \sigma(a) \leq \sigma(a)$*,* $\sigma(a) \odot \top \leq \sigma(a)$*, (iii)* $\sigma(\bot) = \bot$*.*

Proof (i) and (iii) are obvious. We show (ii) $\top \odot \sigma(a) \leq \sigma(a)$.

$$\top \odot \sigma(a) = \top \odot a \cup \top \odot \top \odot a \cup \top \odot a \odot \top \cup \top \odot \top \odot a \odot \top \leq$$
$$\leq \top \odot a \cup \top \odot a \cup \top \odot a \odot \top \cup \top \odot a \odot \top = \top \odot a \cup \top \odot a \odot \top \leq \sigma(a)$$

The proof of $\sigma(a) \odot \top \leq \sigma(a)$ is similar. □

By BSG we denote the class of *boolean semigroups*, i.e. boolean algebras with an associative operation $\odot$ which distributes over $\cup$ in both arguments. Notice that $\sigma(x)$ is a term in the language of BSG. Lemma 4(iii), however, needs $\bot \odot \top = \bot = \top \odot \bot$, which is valid in BRSG, but not in BSG. For $\mathbf{A} \in$ BSG and $c \in A$, we define a map $g_c : A \mapsto A$ as follows: $g_c(x) = x \cup \sigma(c)$ for $x \in A$. On the set $B_c = g_c[A]$ we define an operation $\odot_c$: $a \odot_c b = a \odot b \cup \sigma(c)$.

Lemma 5 *The map* g_c *is an epimorphism of* $(A, \odot)$ *onto* $(B_c, \odot_c)$*.*

Proof $g_c(a) \odot_c g_c(b) = (a \cup \sigma(c)) \odot (b \cup \sigma(c)) \cup \sigma(c) = a \odot b \cup a \odot \sigma(c) \cup \sigma(c) \odot b \cup \sigma(c) \odot \sigma(c) \cup \sigma(c)$. We have $a \odot \sigma(c) \leq \top \odot \sigma(c) \leq \sigma(c)$, by Lemma 4. Similarly $\sigma(c) \odot b \leq \sigma(c)$ and $\sigma(c) \odot \sigma(c) \leq \sigma(c)$. Consequently, the right-hand side of the second equation equals $a \odot b \cup \sigma(c)$, i.e. $g_c(a \odot b)$. □

Corollary 1 $(B_c, \odot_c)$ *is a semigroup.*

In boolean algebras one defines: $a - b = a \cap b^-$, $a \dot{-} b = (a - b) \cup (b - a)$ (symmetric difference). We need the following properties.

$$a = b \Leftrightarrow a \dot{-} b = \bot \tag{14}$$

$$a \cup (a \dot{-} b) = a \cup b = b \cup (a \dot{-} b) \tag{15}$$

For a quasi equation $q := s_1 = t_1 \wedge \cdots \wedge s_n = t_n \Rightarrow s_0 = t_0$ (in language of SG) we define a term t_q and an equation $e(q)$ as follows.

$$t_q := (s_1 \dot{-} t_1) \cup \cdots \cup (s_n \dot{-} t_n) \quad e(q) := s_0 \cup \sigma(t_q) = t_0 \cup \sigma(t_q)$$

Lemma 6 *For any quasi-equation q (in language of SG),* $q \in$ *Queq(SG) if and only if* $e(q) \in$ *Eq(BRSG).*

Proof We prove the if-part. Assume $e(q) \in$ Eq(BRSG). We show $q \in$ Queq(BRSG). Let $\mathbf{A} \in$ BRSG. Let a valuation μ in $\mathbf{A}$ be such that $\mu(s_i) = \mu(t_i)$ for all $i = 1, \ldots, n$. Then, $\mu(t_q) = \bot$, hence $\mu(\sigma(t_q)) = \bot$, by Lemma 4(iii). Consequently, $\mu(s_0) =$

$\mu(s_0 \cup \sigma(t_q)) = \mu(t_0 \cup \sigma(t_q)) = \mu(t_0)$, where the second equation holds, since $e(q)$ is valid in BRSG. So q is valid in BRSG, hence in SG, by Lemma 3.

We prove the only-if part. Assume $e(q) \notin$ Eq(BRSG). There exist $\mathbf{A} \in$ BRSG and a valuation μ in $\mathbf{A}$ such that $\mu(s_0 \cup \sigma(t_q)) \neq \mu(t_0 \cup \sigma(t_q))$. For $c = \mu(t_q)$, we consider the semigroup $(B_c, \odot_c)$ and the epimorphism g_c of $(A, \odot)$ onto $(B_c, , \odot_c)$, defined above.

We show $g_c(\mu(s_i)) = g_c(\mu(t_i))$, for all $i = 1, \ldots, n$. We denote $a_i = \mu(s_i)$, $b_i = \mu(t_i)$ for $i = 0, 1, \ldots, n$. By (15), $s_i \cup t_q = t_i \cup t_q$ is valid in BRSG, for any $i = 1, \ldots, n$. By Lemma 4(i), $t_q \leq \sigma(t_q)$ is valid as well. This yields: $g_c(a_i) = a_i \cup \sigma(c) = a_i \cup c \cup \sigma(c) = b_i \cup c \cup \sigma(c) = b_i \cup c = g_c(b_i)$. On the other hand, $g_c(a_0) = a_0 \cup \sigma(c) \neq b_0 \cup \sigma(c) = g_c(b_0)$, Therefore q is not true in $(B_c, \odot_c)$ for the valuation $g_c \circ \mu$. So $q \notin$ Queq(SG). □

Theorem 5 *[28]* **L-CL** *is undecidable.*

Proof By Lemma 6, Eq(BRSG) is undecidable. This implies the undecidability of **L-CL** (see Remark 4). □

In the proof of Lemma 6, Eq(BRSG) can be replaced with Eq(BSG). Indeed, Lemma 4(iii) is used in the if-part only. With BSG, we assume $e(q) \in$ Eq(BSG), which yields $e(q) \in$ Eq(BRSG). So we can continue as above. Consequently, Eq(BSG) is undecidable [29].

Now we point out the error in [29]. In this paper $e(q) := s_0 \dot{-} t_0 \leq \sigma(t_q)$. The argument for the only-if part of Lemma 6 does not work. Assuming $e(q) \notin$ Eq(BSG), one obtains $\mu(s_0 \dot{-} t_0) \neq \bot$, hence $\mu(s_0) \neq \mu(t_0)$. This, however, does not imply $g_c(\mu(s_0)) \neq g_c(\mu(t_0))$ (claimed in [29]), since g_c need not be a monomorphism. In our proof, this inequality, written $g_c(a_0) \neq g_c(b_0)$, holds by the form of $e(q)$. For honesty, let us note that in [28] $e(q)$ is defined differently: $(s_0 \cup \sigma(t_q)) \dot{-} (t_0 \cup \sigma(t_q)) \leq \sigma(t_q)$ (in fact, the construction is more complicated, but it takes this form if a noncommutative operation, used in [28], is replaced with $\dot{-}$). This works!

In the same way one proves the undecidability of **L1-CL** (not claimed in [28, 29]). It suffices to note that the word problem for monoids is also undecidable. By dual constructions, one proves the undecidability of intuitionistic logic with an associative binary modality which distributes over $\wedge$; we denote it by $\bullet$. This is briefly noted in [28] without any details, but it is not difficult to recover them. Intuitionistic $\leftrightarrow$ replaces $\dot{-}$. Our $\sigma(x)$ is replaced by $\delta(x) := x \cap \bot \bullet x \cap x \bullet \bot \cap \bot \bullet x \bullet \bot$. Finally:

$$t_q := (s_1 \leftrightarrow t_1) \cap \cdots \cap (s_n \leftrightarrow t_n) \quad e(q) := s_0 \cap \delta(t_q) = t_0 \cap \delta(t_q)$$

By dualizing the arguments, written above, one can prove the undecidability of $\mathbf{L}^d$-**IL**, i.e. intuitionistic logic augmented with dual Lambek connectives $\bullet, \backslash^\bullet, /^\bullet$, with the axioms (id), (a1), (a2) (for $\bullet$), the rules corresponding to (RES$\bullet$) and (cut-1). Interestingly, **L-IL** is decidable [22]: the proof employs a cut-free sequent system for this logic.

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