Full Nonassociative Lambek Calculus with Distribution: Models and Grammars

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Abstract

We study Nonassociative Lambek Calculus with additives $\wedge, \vee$, satisfying the distributive law (Full Nonassociative Lambek Calculus with Distribution **DFNL**). We prove that formal grammars based on **DFNL**, also with assumptions, generate context-free languages. The proof uses proof-theoretic tools (interpolation) and a construction of a finite model, employed in [13] in the proof of Strong Finite Model Property of **DFNL**. We obtain analogous results for different variants of **DFNL**, e.g. **BFNL**, which admits negation $\neg$ such that $\wedge, \vee, \neg$ satisfy the laws of boolean algebra, and **HFNL** whose underlying lattice is a Heyting algebra. Our proof also yields Finite Embeddability Property for boolean and Heyting algebras, supplied with an additional residuation structure.

1 Introduction

Nonassociative Lambek Calculus **NL** proves the order formulas $\alpha \leq \beta$, valid in *residuated groupoids*, i.e. ordered algebras $(M, \cdot, \backslash, /, \leq)$ such that $(M, \leq)$ is a poset, and $\cdot, \backslash, /$ are binary operations on M, satisfying the residuation law:

$$a \cdot b \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b\,, \tag{1}$$

for all $a, b, c \in M$. As an easy consequence of (1), we obtain:

$$a(a\backslash b) \leq b\,,\ (a/b)b \leq a\,, \tag{2}$$

$$\text{if } a \leq b \text{ and } c \leq d \text{ then } ac \leq bd,\ b\backslash c \leq a\backslash d,\ c/b \leq d/a\,, \tag{3}$$

for all $a, b, c \in M$. Hence every residuated groupoid is a partially ordered groupoid, if one forgets residuals $\backslash, /$ (we refer to $\cdot$ as product).

NL was introduced by Lambek [20] as a variant of Syntactic Calculus [19], now called Associative Lambek Calculus **L**, which yields the order formulas valid in residuated semigroups ($\cdot$ is associative). Both are standard type logics for categorial grammars [3, 10, 23, 24]. While **L** is appropriate for expressions in the form of strings, **NL** corresponds to tree structures. The cut-elimination theorem holds for **NL** and **L**, and it yields the decidability of these systems [19, 20].

NL and **L** are examples of *substructural logics*, i.e. non-classical logics whose sequent systems lack some structural rules (Weakening, Contraction, Exchange); see [14]. Besides multiplicatives $\cdot, \backslash, /$, substructural logics usually admit additives $\wedge, \vee$. The corresponding algebras are residuated lattices $(M, \wedge, \vee, \cdot, \backslash, /, 1)$: here $(M, \wedge, \vee)$ is a lattice, $(M, \cdot, 1)$ is a monoid (i.e. a semigroup with 1), and (1) holds. For the nonassociative case, monoids are replaced by groupoids or unital groupoids (i.e. groupoids with 1); the resulting algebras are called lattice-ordered residuated (unital) groupoids. An algebra of that kind is said to be distributive, if its lattice reduct is distributive. The complete logic for residuated lattices is Full Lambek Calculus **FL** [14]. Its sequent system admits cut-elimination; **FL** is decidable [26]. Residuated lattices form a variety, and so for lattice-ordered residuated groupoids, but it is not true for residuated semigroups, nor residuated groupoids [14].

Categorial grammars based on **NL** generate precisely the $\epsilon-$free context-free languages [7, 18]. Pentus [27] proves the same for **L**. Using **FL**, even without $\vee$, one can generate languages which are not context-free, i.e. meets of two context-free languages [17]. This also holds for **FL** with distribution, since it is conservative over its $\vee-$free fragment. The provability problem for **L** is NP-complete [28]; for **FL**, the upper bound is P-SPACE.

Full Nonassociative Lambek Calculus **FNL** is the complete logic of lattice-ordered residuated groupoids. We present it in the form of a sequent system. The cut-elimination theorem holds for this system [14]. It is not useful for our purposes, since we consider the consequence relation of **FNL** which requires the cut rule to be complete with respect to algebraic models. Furthermore, our main issue is **DFNL**, and the distributive law is affixed to **FNL** as a new axiom; the cut rule is necessary in **DFNL**.

We prove that (in opposition to **FL**) categorial grammars based on **DFNL** generate context-free languages, and it remains true if one adds an arbitrary finite set of assumptions to **DFNL**. For **NL**, an analogous result has been proved in [11]. The latter paper also proves the polytime decidability of the consequence relation of **NL** ([15] makes it for provability in **NL**), but it cannot be shown in the presence of additives. The consequence relation of **DFNL** is decidable, which follows from Strong Finite Model Property (SFMP), proved in [13], employing some ideas of [25, 2, 11]. (The consequence relations for **L**, **FL** are undecidable [6, 14].)

The construction of a finite model, used in the proof of SFMP for **DFNL** in [13], will also be employed here (in a modified form) in order to prove an interpolation lemma, needed for context-freeness. Our methods can be extended to multi-modal variants of **DFNL** which admit several 'product' operations (of arbitrary arity) and the corresponding residual operations. Without additives, this multi-modal framework was presented in [8, 10, 11]. (It is also naturally related to multi-modal extensions of Lambek Calculus, studied in e.g. [12, 23, 24].) This leads us to the proof of SFMP for **BFNL**, which is the complete logic of boolean-ordered residuated groupoids, and the context-freeness of the corresponding grammars, and the same holds for **HFNL**, which is the complete logic of Heyting algebras with an additional residuation structure. These results can be generalized to multi-modal systems. All classes of algebras, considered here, are closed under products (and contain the trivial algebra, i.e. the empty product), and consequently, SFMP implies Finite Embeddability Property: every finite partial subalgebra of an algebra from this class can be embedded in a finite algebra from this class (this is equivalent to FMP of the universal theory of the class of algebras).

Distribution is essential for the construction of a finite model in the proof of SFMP and the finiteness of the set of possible interpolants in our interpolation lemma. So, our results cannot easily be adapted for systems without distribution.

(External) consequence relations for substructural logics have been studied in different contexts; see e.g. [1, 5, 14]. Put it differently, one studies logics enriched with (finitely many) assumptions. Assumptions are sequents (not closed under substitution) added to axioms of the system (with the cut rule). Categorial grammars are usually required to be lexical in the sense that the logic is common for all languages and all information on the particular language is contained in the type lexicon. But, there are approaches allowing non-lexical assumptions, which results in a more efficient description of the language and an increase of generative power [6, 10, 21, 22]. Let us emphasize that our results on context-freeness are new even for pure logics **DFNL**, **BFNL** and their variants, and assumptions do not change anything essential in proofs.

2 Restricted interpolation

We admit a denumerable set of variables $p, q, r, \ldots$. Formulas are built from variables by means of $\cdot, \backslash, /, \wedge, \vee$. Formula structures (shortly: structures) are built from formulas according to the rule: if X, Y are structures then (X, Y) is a structure. We denote arbitrary formulas by $\alpha, \beta, \gamma, \ldots$ and structures by X, Y, Z. $X[Y]$ denotes a structure X with a designated substructure Y; in this context, $X[Z]$ denotes the substitution of Z for Y in X.

Sequents are of the form $X \Rightarrow \alpha$. **FNL** assumes the following axioms and inference rules:

$$(\mathrm{Id})\ \alpha \Rightarrow \alpha\,,$$

$$(\cdot\mathrm{L})\ \frac{X[(\alpha,\beta)] \Rightarrow \gamma}{X[\alpha\cdot\beta] \Rightarrow \gamma},\ (\cdot\mathrm{R})\ \frac{X \Rightarrow \alpha;\ Y \Rightarrow \beta}{(X,Y) \Rightarrow \alpha\cdot\beta},$$

$$(\backslash\mathrm{L})\ \frac{X[\beta] \Rightarrow \gamma;\ Y \Rightarrow \alpha}{X[(Y,\alpha\backslash\beta)] \Rightarrow \gamma},\ (\backslash\mathrm{R})\ \frac{(\alpha,X) \Rightarrow \beta}{X \Rightarrow \alpha\backslash\beta},$$

$$(/\mathrm{L})\ \frac{X[\beta] \Rightarrow \gamma;\ Y \Rightarrow \alpha}{X[(\beta/\alpha,Y)] \Rightarrow \gamma},\ (/\mathrm{R})\ \frac{(X,\alpha) \Rightarrow \beta}{X \Rightarrow \beta/\alpha},$$

$$(\wedge\mathrm{L})\ \frac{X[\alpha_i] \Rightarrow \beta}{X[\alpha_1 \wedge \alpha_2] \Rightarrow \beta},\ (\wedge\mathrm{R})\ \frac{X \Rightarrow \alpha;\ X \Rightarrow \beta}{X \Rightarrow \alpha \wedge \beta},$$

$$(\vee\mathrm{L})\ \frac{X[\alpha] \Rightarrow \gamma;\ X[\beta] \Rightarrow \gamma}{X[\alpha \vee \beta] \Rightarrow \gamma},\ (\vee\mathrm{R})\ \frac{X \Rightarrow \alpha_i}{X \Rightarrow \alpha_1 \vee \alpha_2},$$

$$(\mathrm{CUT})\ \frac{X[\alpha] \Rightarrow \beta;\ Y \Rightarrow \alpha}{X[Y] \Rightarrow \beta}.$$

In ($\wedge$L) and ($\vee$R), the subscript i equals 1 or 2. The latter rules and ($\cdot$L), (\R), (/R) have one premise; the remaining rules have two premises, separated by semicolon. **DFNL** admits the additional axiom scheme:

$$(\mathrm{D})\ \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\,.$$

(CUT) can be eliminated from **FNL** but not from **DFNL**. Let Φ be a set of sequents. We write $\Phi \vdash X \Rightarrow \alpha$ if $X \Rightarrow \alpha$ is deducible from Φ in **DFNL**. By $F(X)$ we denote the formula arising from X after one has replaced each comma by $\cdot$. By ($\cdot$L) and (Id), ($\cdot$R), (CUT), $X \Rightarrow \alpha$ and $F(X) \Rightarrow \alpha$ are mutually deducible. Consequently, without loss of generality we can assume that Φ consists of sequents of the form $\alpha \Rightarrow \beta$ (simple sequents). In models, $\Rightarrow$ is interpreted as $\leq$ and, by definition, an assignment f satisfies $X \Rightarrow \alpha$ iff $f(F(X)) \leq f(\alpha)$.

In what follows, we always assume that Φ is a finite set of simple sequents. T denotes a set of formulas. By a $T-$sequent we mean a sequent such that all formulas occurring in it belong to T. We write $X \Rightarrow_T \alpha$ if $X \Rightarrow \alpha$ has a deduction from Φ in **DFNL** which consists of $T-$sequents only (then, $X \Rightarrow \alpha$ must be a $T-$sequent). The following lemma is proved for **DFNL** but the same proof works for **FNL**.

Lemma 1. *Let T be closed under $\wedge, \vee$. Let $X[Y] \Rightarrow_T \gamma$. Then, there exists $\delta \in T$ such that $X[\delta] \Rightarrow_T \gamma$ and $Y \Rightarrow_T \delta$.*

Proof. δ is called an interpolant of Y in $X[Y] \Rightarrow \gamma$. The proof proceeds by induction on $T-$deductions of $X[Y] \Rightarrow \gamma$ from Φ. The case of axioms and assumptions is easy; they are simple sequents $\alpha \Rightarrow \gamma$, so $Y = \alpha$ and $\delta = \alpha$.

Let $X[Y] \Rightarrow \gamma$ be the conclusion of a rule. (CUT) is easy. If Y comes from one premise of (CUT), then we take an interpolant from this premise. Otherwise Y must contain Z, where the premises are $X[\alpha] \Rightarrow \gamma$, $Z \Rightarrow \alpha$. So, $Y = U[Z]$, and it comes from $U[\alpha]$ in the first premise. Then, an interpolant δ of $U[\alpha]$ in this premise is also an interpolant of Y in the conclusion, by (CUT).

Let us consider other rules. First, we assume that Y does not contain the formula, introduced by the rule (the active formula). If Y comes from exactly one premise of the rule, then one takes an interpolant from this premise. Let us consider ($\wedge$R). The premises are $X[Y] \Rightarrow \alpha$, $X[Y] \Rightarrow \beta$, and the conclusion is $X[Y] \Rightarrow \alpha \wedge \beta$. By the induction hypothesis, there are interpolants δ of Y in the first premise and δ' of Y in the second one. We have $X[\delta] \Rightarrow_T \alpha$, $X[\delta'] \Rightarrow_T \beta$, $Y \Rightarrow_T \delta$, $Y \Rightarrow_T \delta'$. Then, $\delta \wedge \delta'$ is an interpolant of Y in the conclusion, by ($\wedge$L), ($\wedge$R). Let us consider ($\vee$L). The premises are $X[\alpha][Y] \Rightarrow \gamma$, $X[\beta][Y] \Rightarrow \gamma$, and the conclusion is $X[\alpha \vee \beta][Y] \Rightarrow \gamma$, where Y does not contain $\alpha \vee \beta$. As above, there are interpolants δ, δ' of Y in the premises. Again $\delta \wedge \delta'$ is an interpolant of Y in the conclusion, by ($\wedge$L), ($\vee$L) and ($\wedge$R). For ($\cdot$R) with premises $U \Rightarrow \alpha$, $V \Rightarrow \beta$ and conclusion $(U, V) \Rightarrow \alpha \cdot \beta$, if $Y = (U, V)$, then we take $\delta = \alpha \cdot \beta$.

Second, we assume that Y contains the active formula (so, the rule must be an L-rule). If Y is a single formula, then we take $\delta = Y$. Assume that Y is not a formula. For ($\cdot$L), ($\wedge$L), we take an interpolant of Y' in the premise, where Y' is the natural source of Y. For ($\backslash$L) with premises $X[\beta] \Rightarrow \gamma$, $Z \Rightarrow \alpha$ and conclusion $X[(Z, \alpha\backslash\beta)] \Rightarrow \gamma$, we consider the source Y' of Y (Y' occurs in $X[\beta]$ and contains β). Then, Y arises from Y' by substituting $(Z, \alpha\backslash\beta)$ for β. Hence, an interpolant of Y' in the first premise is also an interpolant of Y in the conclusion, by ($\backslash$L). The case of (/L) is similar. The final case is ($\vee$L) with premises $Z[U[\alpha]] \Rightarrow \gamma$, $Z[U[\beta]] \Rightarrow \gamma$ and conclusion $Z[U[\alpha \vee \beta]] \Rightarrow \gamma$, where $Y = U[\alpha \vee \beta]$. Let δ be an interpolant of $U[\alpha]$ in the first premise and δ' be an interpolant of $U[\beta]$ in the second premise. Then, $\delta \vee \delta'$ is an interpolant of Y in the conclusion, by ($\vee$L), ($\vee$R). $\square$

3 Finite models and interpolation

We prove an (extended) subformula property and an interpolation lemma for the deducibility relation $\vdash$ in **DFNL**. We need some constructions of lattice-ordered residuated groupoids.

Let $(M, \cdot)$ be a groupoid. On the powerset $P(M)$ one defines operations: $U \cdot V = \{ab : a \in U, b \in V\}$, $U\backslash V = \{c \in M : U \cdot \{c\} \subseteq V\}$, $U/V = \{c \in M : \{c\} \cdot V \subseteq U\}$, $U \vee V = U \cup V$, $U \wedge V = U \cap V$. $P(M)$ with these operations is a distributive lattice-ordered groupoid (it is a complete lattice). The order is $\subseteq$.

An operator $C : P(M) \mapsto P(M)$ is called a closure operator on $(M, \cdot)$, if it satisfies the following conditions: (C1) $U \subseteq C(U)$, (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$, (C3) $C(C(U)) \subseteq U$, (C4) $C(U) \cdot C(V) \subseteq C(U \cdot V)$, for all $U, V \subseteq M$ [14]. A set $U \subseteq M$ is said to be closed, if $C(U) = U$. By $C(M, \cdot)$ we denote the family of all closed subsets of M. Operations on $C(M, \cdot)$ are defined as follows: $U \otimes V = C(U \cdot V)$, $U\backslash V$, U/V and $U \wedge V$ as above, $U \vee V = C(U \cup V)$. (The product operation in $C(M, \cdot)$ is denoted by $\otimes$ to avoid collision with $\cdot$ in $P(M)$.) It is known that $C(M, \cdot)$ with these operations is a complete lattice-ordered residuated groupoid [14]; it need not be distributive. The order is $\subseteq$.

(C4) is essential in the proof that $C(M, \cdot)$ is closed under $\backslash, /$. Actually, if U is closed, then $V\backslash U$ and U/V are closed, for any $V \subseteq M$. Let us consider U/V. Since (2) hold in $P(M)$, then $(U/V) \cdot V \subseteq U$. We get $C(U/V) \cdot V \subseteq C(U/V) \cdot C(V) \subseteq C((U/V) \cdot V) \subseteq C(U) = U$, and consequently, $C(U/V) \subseteq U/V$, by (1) for $P(M)$. The reader is invited to prove that (1) holds in $C(M, \cdot)$ and $C(U \cup V)$ is the join of U, V in $C(M, \cdot)$.

We consider extended formula structures which may contain a special atomic substructure $\circ$. Contexts are extended structures which contain a unique occurrence of $\circ$. If $Z[\circ]$ is a context and X is a structure, then $Z[X]$ denotes the substitution of X for $\circ$ in $Z[\circ]$.

Let T be a nonempty set of formulas. By T^* we denote the set of all structures formed out of formulas from T. $T^*[\circ]$ denotes the set of all contexts whose all atomic substructures different from $\circ$ belong to T.

T^* is a (free) groupoid with the operation $X \cdot Y = (X, Y)$. Hence $P(T^*)$ is a lattice-ordered residuated groupoid with operations defined as above. For $Z[\circ] \in T^*[\circ]$ and $\alpha \in T$, we define a set:

$$[Z[\circ], \alpha] = \{X \in T^* : Z[X] \Rightarrow_T \alpha\}. \tag{4}$$

The family of all sets $[Z[\circ], \alpha]$, defined in this way, is denoted $B(T)$. An operator $C_T : P(T^*) \mapsto P(T^*)$ is defined as follows:

$$C_T(U) = \bigcap\{[Z[\circ], \alpha] \in B(T) : U \subseteq [Z[\circ], \alpha]\}\,, \tag{5}$$

for $U \subseteq T^*$. It is easy to see that C_T satisfies (C1), (C2), (C3). We prove (C4). Let $U, V \subseteq T^*$ and $X \in C_T(U)$, $Y \in C_T(V)$. We show $(X, Y) \in C_T(U \cdot V)$. Let $[Z[\circ], \alpha] \in B(T)$ be such that $U \cdot V \subseteq [Z[\circ], \alpha]$. For any $X' \in U$, $Y' \in V$, $(X', Y') \in [Z[\circ], \alpha]$, whence $Z[(X', Y')] \Rightarrow_T \alpha$. So, $U \subseteq [Z[(\circ, Y')], \alpha]$, whence $C_T(U) \subseteq [Z[(\circ, Y')], \alpha]$, by (5), and the latter holds for any $Y' \in V$. Then, $Z[(X, Y')] \Rightarrow_T \alpha$, for any $Y' \in V$. We get $V \subseteq [Z[(X, \circ)], \alpha]$, which yields $C_T(V) \subseteq [Z[(X, \circ)], \alpha]$, by (5). Consequently, $Z[(X, Y)] \Rightarrow_T \alpha$, whence $(X, Y) \in [Z[\circ], \alpha]$ (see [25, 2, 14, 13] for similar arguments).

We have shown that C_T is a closure operator on $(T^*, \cdot)$. We consider the algebra $C_T(T^*, \cdot)$, further denoted by $M(T, \Phi)$. Clearly, all sets in $B(T)$ are closed under C_T. We define:

$$[\alpha] = [\circ, \alpha] = \{X \in T^*; X \Rightarrow_T \alpha\}\,. \tag{6}$$

For $\alpha \in T$, $[\alpha] \in B(T)$. The following equations are true in $M(T, \Phi)$ provided that all formulas appearing in them belong to T.

$$[\alpha] \otimes [\beta] = [\alpha \cdot \beta],\ [\alpha]\backslash[\beta] = [\alpha\backslash\beta],\ [\alpha]/[\beta] = [\alpha/\beta]\,, \tag{7}$$

$$[\alpha] \vee [\beta] = [\alpha \vee \beta],\ [\alpha] \wedge [\beta] = [\alpha \wedge \beta]\,. \tag{8}$$

We prove the first equation (7). If $X \Rightarrow_T \alpha$ and $Y \Rightarrow_T \beta$ then $(X, Y) \Rightarrow_T \alpha{\cdot}\beta$, by $(\cdot\mathrm{R})$. Consequently, $[\alpha]{\cdot}[\beta] \subseteq [\alpha{\cdot}\beta]$. Then $[\alpha]{\otimes}[\beta] = C_T([\alpha]{\cdot}[\beta]) \subseteq [\alpha{\cdot}\beta]$, by (C2), (C3). We prove the converse inclusion. Let $[Z[\circ], \gamma] \in B(T)$ be such that $[\alpha] \cdot [\beta] \subseteq [Z[\circ], \gamma]$. By (Id), $\alpha \in [\alpha]$, $\beta \in [\beta]$, whence $Z[(\alpha, \beta)] \Rightarrow_T \gamma$. Then, $Z[\alpha \cdot \beta] \Rightarrow_T \gamma$, by $(\cdot\mathrm{R})$. Hence, if $X \in [\alpha \cdot \beta]$ then $Z[X] \Rightarrow_T \gamma$, by (CUT), which yields $X \in [Z[\circ], \gamma]$. We have shown $[\alpha \cdot \beta] \subseteq C_T([\alpha] \cdot [\beta])$.

We prove the second equation (7). Let $X \in [\alpha]\backslash[\beta]$. Since $\alpha \in [\alpha]$, then $(\alpha, X) \in [\beta]$. Hence $(\alpha, X) \Rightarrow_T \beta$, which yields $X \Rightarrow_T \alpha\backslash\beta$, by $(\backslash\mathrm{R})$. We have shown $\subseteq$. To prove the converse inclusion it suffices to show $[\alpha] \cdot [\alpha\backslash\beta] \subseteq [\beta]$. If $X \Rightarrow_T \alpha$ and $Y \Rightarrow_T \alpha\backslash\beta$, then $(X, Y) \Rightarrow_T \beta$, since $(\alpha, \alpha\backslash\beta) \Rightarrow_T \beta$, by (Id), $(\backslash\mathrm{L})$, and one applies (CUT). The proof of the third equation (7) is similar. Proofs of (8) are left to the reader.

We say that formulas $\alpha, \beta \in T$ are $T-$equivalent, if $\alpha \Rightarrow_T \beta$ and $\beta \Rightarrow_T \alpha$. By (Id), (CUT), $T-$equivalence is an equivalence relation. By $\overline{T}$ we denote the smallest set of formulas which contains all formulas from T and is closed under subformulas and $\wedge, \vee$. If T is closed under subformulas, then $\overline{T}$ is the closure of T under $\wedge, \vee$.

Lemma 2. *If T is a finite set of formulas, then $\overline{T}$ is finite up to $\overline{T}-$equivalence.*

Proof. If T is finite, then the set T' of subformulas of formulas from T is also finite. $\overline{T}$ is the closure of T' under $\wedge, \vee$. The converse of (D) $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \Rightarrow \alpha \wedge (\beta \vee \gamma)$ is provable in **FNL** (it is valid in all lattices); if $\alpha, \beta, \gamma \in \overline{T}$, then the proof uses $\overline{T}-$sequents only. Consequently, for $\alpha, \beta, \gamma \in \overline{T}$, both sides of (D) are $\overline{T}-$equivalent. It follows that every formula from $\overline{T}$ is $\overline{T}-$equivalent to a finite disjunction of finite conjunctions of formulas from T'. If one omits repetitions, then there are only finitely many formulas of the latter form. $\square$

Recall that an assignment in a model M is a homomorphism from the formula algebra into M.

Lemma 3. *Let T be a nonempty, finite set of formulas. Then, $M(\overline{T}, \Phi)$ is a finite distributive lattice-ordered residuated groupoid. For any assignment f in $M(\overline{T}, \Phi)$ such that $f(p) = [p]$, for any $p \in \overline{T}$, and any $\overline{T}-$sequent $X \Rightarrow \alpha$, f satisfies $X \Rightarrow \alpha$ in $M(\overline{T}, \Phi)$ if and only if $X \Rightarrow_{\overline{T}} \alpha$.*

Proof. As shown above, $M(\overline{T}, \Phi)$ is a lattice-ordered residuated groupoid. We prove the second part of the lemma. Let f satisfy $f(p) = [p]$, for any variable $p \in \overline{T}$. Using (7), (8), one proves $f(\alpha) = [\alpha]$, for all $\alpha \in \overline{T}$, by easy formula induction.

Assume that f satisfies the $\overline{T}-$sequent $X \Rightarrow \alpha$. For any formula β appearing in X, we have $\beta \in [\beta] = f(\beta)$, whence $X \in f(F(X))$. Since $f(F(X)) \subseteq f(\alpha)$, then $X \in f(\alpha) = [\alpha]$. Thus $X \Rightarrow_{\overline{T}} \alpha$. Assume $X \Rightarrow_{\overline{T}} \alpha$. Then, there exists a $\overline{T}-$deduction of $X \Rightarrow \alpha$ from Φ in **DFNL**. By induction on this deduction, we prove that f satisfies $X \Rightarrow \alpha$ in $M(\overline{T}, \Phi)$. f obviously satisfies axioms (Id). Assumptions from Φ and instances of (D), restricted to $\overline{T}-$sequents, are of the form $\beta \Rightarrow \gamma$, where $\beta, \gamma \in \overline{T}$. Since $\beta \Rightarrow_{\overline{T}} \gamma$, then $[\beta] \subseteq [\gamma]$, by (CUT), which yields $f(\beta) \subseteq f(\gamma)$. The rules of **FNL** are sound for any assignment in a lattice-ordered residuated groupoid, which finishes this part of proof.

Let R be a selector of the family of equivalence classes of $\overline{T}-$equivalence (R chooses one formula from each equivalence class). By Lemma 2, R is a nonempty finite subset of $\overline{T}$. We show that every nontrivial (i.e. nonempty and not total) closed subset of $\overline{T}^*$ equals $[\alpha]$, for some $\alpha \in R$. Let U be nontrivial and closed.. Let $X \in U$. There exists a set $[Z[\circ], \beta] \in B(\overline{T})$ such that $U \subseteq [Z[\circ], \beta]$. So, $Z[X] \Rightarrow_{\overline{T}} \beta$. By Lemma 1, there exists $\delta \in \overline{T}$ such that $Z[\delta] \Rightarrow_{\overline{T}} \beta$ and $X \Rightarrow_{\overline{T}} \delta$. We get $X \in [\delta]$ and $[\delta] \subseteq [Z[\circ], \beta]$, by (CUT). Clearly we can take $\delta \in R$. We can find such a formula $\delta \in R$, for any set $[Z[\circ], \beta] \in B(\overline{T})$ such that $U \subseteq [Z[\circ], \beta]$. Thus, we obtain a nonempty finite set $S \subseteq R$ such that, for any $[Z[\circ], \beta] \in B(\overline{T})$ such that $U \subseteq [Z[\circ], \beta]$, there exists $\delta \in S$ such that $X \in [\delta]$ and $[\delta] \subseteq [Z[\circ], \beta]$. Let γ_X be the conjunction of all formulas from S. By (8), (C3) and (6), $X \in [\gamma_X]$

and $[\gamma_X] \subseteq U$. Again we can replace γ_X by a $\overline{T}-$equivalent formula from R. So, we stipulate $\gamma_X \in R$. Let α be the disjunction of all formulas γ_X, for $X \in U$ (there are only finitely many different formulas of that kind). By (8), $[\alpha] \subseteq U$ and, evidently, $U \subseteq [\alpha]$. We can stipulate $\alpha \in R$.

It follows that $M(\overline{T}, \Phi)$ is finite. We prove that it is distributive. It suffices to prove $U \wedge (V \vee W) \subseteq (U \wedge V) \vee (U \wedge W)$, for any closed sets U, V, W. This inclusion is true, if at least one of the sets U, V, W is empty or total, since $M(\overline{T}, \Phi)$ is a lattice. So, assume U, V, W be nontrivial. By the above paragraph, $U = [\alpha]$, $V = [\beta]$, $W = [\gamma]$, for some $\alpha, \beta, \gamma \in R$. Then, the inclusion follows from (8) and the fact that $[\alpha \wedge (\beta \vee \gamma)] \subseteq [(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)]$. □

Notice that Lemma 3 implies the decidability of $\vdash$, since it yields SFMP: if $\Phi \vdash X \Rightarrow \alpha$ does not hold, then there exist a finite model M and an assignment f in M such that f satisfies Φ but does not satisfy $X \Rightarrow \alpha$. We are ready to prove an extended subformula property and an interpolation lemma for **DFNL**.

Lemma 4. *Let T be a finite set of formulas, containing all formulas appearing in $X \Rightarrow \alpha$ and Φ. If $\Phi \vdash X \Rightarrow \alpha$ then $X \Rightarrow_{\overline{T}} \alpha$.*

Proof. Let f be an asignment in $M(\overline{T}, \Phi)$, satisfying $f(p) = [p]$, for any variable $p \in \overline{T}$. Let $\beta \Rightarrow \gamma$ be a sequent from Φ. Then $\beta \Rightarrow_{\overline{T}} \gamma$, which yields $f(\beta) \subseteq f(\gamma)$, by Lemma 3. So, f satisfies all sequents from Φ.

Assume $\Phi \vdash X \Rightarrow \alpha$. Since **DFNL** is strongly sound with respect to distributive lattice-ordered residuated groupoids, then f satisfies $X \Rightarrow \alpha$. Consequently, $X \Rightarrow_{\overline{T}} \alpha$, by Lemma 3. □

Lemma 5. *Let T be a finite set of formulas, containing all formulas appearing in $X[Y] \Rightarrow \alpha$ and Φ. If $\Phi \vdash X[Y] \Rightarrow \alpha$ then there exists $\delta \in \overline{T}$ such that $\Phi \vdash X[\delta] \Rightarrow \alpha$ and $\Phi \vdash Y \Rightarrow \delta$.*

Proof. Assume $\Phi \vdash X[Y] \Rightarrow \alpha$. By Lemma 4, $X[Y] \Rightarrow_{\overline{T}} \alpha$. Apply Lemma 1. □

Lemma 3 except for the finiteness and distributivity of $M(\overline{T}, \Phi)$ and Lemmas 4 and 5 also hold for **FNL**. For **NL**, Lemma 4 and Lemma 5 hold with $\overline{T}$ defined as the closure of T under subformulas [11] (for pure **NL**, a weaker form of the latter lemma was proved in [16]).

4 Categorial grammars based on DFNL

A categorial grammar based on a logic $\mathcal{L}$ (presented as a sequent system) is defined as a tuple $G = (\Sigma, I, \alpha, \Phi)$ such that Σ is a finite alphabet, I is a nonempty finite relation between elements of Σ and formulas of $\mathcal{L}$, α is a formula of $\mathcal{L}$, and Φ is a finite set of sequents of $\mathcal{L}$. Elements of Σ are usually interpreted as words from the lexicon of a language and strings on Σ as phrases. Formulas of $\mathcal{L}$ are called types. I assigns finitely many types to each word from Σ. α is a designated type; often one takes a designated variable s (the type of sentences).

$\mathcal{L}$ is the logic of type change and composition. Φ is a finite set of assumptions added to $\mathcal{L}$.

Our logic $\mathcal{L}$ is **DFNL**. Let $G = (\Sigma, I, \alpha, \Phi)$ be a categorial grammar. By $T(G)$ we denote the set of all types appearing in the range of I. Let T be the smallest set containing $T(G)$, all types from Φ and α. For any type β, we define $L(G, \beta) = \{X \in \overline{T}^* : \Phi \vdash X \Rightarrow \beta\}$. Elements of $\overline{T}^*$ can be seen as finite binary trees whose leaves are labeled by types from $\overline{T}$. The tree language of G, denoted by $L_t(G)$, consists of all trees which can be obtained from trees in $L(G, \alpha) \cap T(G)^*$ by replacing each type γ by some $a \in \Sigma$ such that $(a, \gamma) \in I$. The language of G, denoted by $L(G)$, is the yield of $L_t(G)$.

Theorem 1. *$L(G)$ is a context-free language, for any categorial grammar G based on* **DFNL**.

Proof. Fix $G = (\Sigma, I, \alpha, \Phi)$. We define a context-free grammar G' such that $L(G') = L(G)$. Let T be defined as above. By Lemma 2, $\overline{T}$ is finite up to $\overline{T}-$equivalence. We choose a set $R \subseteq \overline{T}$ which contains one formula from each equivalence class. For $\beta \in \overline{T}$, by $r(\beta)$ we denote the unique type from R which is $\overline{T}-$equivalent to β.

G' is defined as follows. The terminal alphabet is Σ. The nonterminal alphabet is R. Production rules are: (R1) $\beta \mapsto \gamma$, for $\beta, \gamma \in R$ such that $\Phi \vdash \gamma \Rightarrow \beta$, (R2) $\beta \mapsto \gamma\delta$, for $\beta, \gamma, \delta \in R$ such that $\Phi \vdash (\gamma, \delta) \Rightarrow \beta$, (R3) $r(\beta) \mapsto a$, for $\beta \in T(G)$, $a \in \Sigma$ such that $(a, \beta) \in I$. The initial symbol is $r(\alpha)$.

Every derivation tree in G' can be treated as a deduction from Φ in **DFNL** which is based on deducible sequents appearing in (R1), (R2) and (CUT). Then, $L(G') \subseteq L(G)$. The converse inclusion follows from Lemma 5. Let $x \in L(G)$. Then, x is the yield of some $Y \in L_t(G)$. There exists $X \in L(G, \alpha)$ such that Y is obtained from X in the way described above. Let $r(X)$ denote the tree resulting from X after one has replaced each type β by $r(\beta)$. Clearly, if $X \in L(G, \gamma)$, then $r(X) \in L(G, r(\gamma)$. It suffices to prove that, for any $\gamma \in \overline{T}$ and any $X \in L(G, \gamma)$, there exists a derivation of $r(X)$ from $r(\gamma)$ (as a derivation tree) in G'. We proceed by induction on the number of commas in X. Let $X \in L(G, \gamma)$ be a single type, say, $X = \beta$. Then, $\Phi \vdash \beta \Rightarrow \gamma$, whence $\Phi \vdash r(\beta) \Rightarrow r(\gamma)$. Then, $r(X) = r(\beta)$ is derivable from $r(\gamma)$, by (R1). Let $X \in L(G, \gamma)$ contain a comma. Then, X must contain a substructure of the form (δ_1, δ_2), where $\delta_i \in \overline{T}$. We write $X = Z[(\delta_1, \delta_2)]$. By Lemma 5, there exists $\delta \in \overline{T}$ such that $\Phi \vdash Z[\delta] \Rightarrow \gamma$ and $(\delta_1, \delta_2) \Rightarrow \delta$. By the induction hypothesis, $r(Z[\delta])$ can be derived from $r(\gamma)$ in G'. Then, $r(X)$ can be derived from $r(\gamma)$, by (R2). $\square$

It has been shown in [7, 18] that every $\epsilon-$free context-free language can be generated by a categorial grammar based on **NL** which uses very restricted types only: p, $p\backslash q$, $p\backslash(q\backslash r)$, where p, q, r are variables; the designated type is also a variable s. Now, we use the fact that **DFNL** is conservative over **NL**, since every residuated groupoid can be embedded into a powerset algebra over a groupoid [10]. Accordingly, every $\epsilon-$free context-free language is generated by some categorial grammar based on **DFNL**.

5 Variants

The methods of this paper cannot be applied to associative systems **L**, **FL**, **FL+D**. Consequence relations for these systems are undecidable; see [6, 11, 14]. Hence no analogue of Lemma 3 can be true. It is also easy to find counterexamples falsifying Lemmas 1, 4 and 5. Analogues of Theorem 1 are false (see Introduction).

They can be applied to several other nonassociative systems. The first example is $\mathbf{DFNL}_e$, i.e. **DFNL** with the exchange rule:

$$\text{(EXC)}\ \frac{X[(Y,Z)] \Rightarrow \alpha}{X[(Z,Y)] \Rightarrow \alpha}.$$

$\mathbf{DFNL}_e$ is complete with respect to distributive lattice-ordered commutative residuated groupoids ($ab = ba$, for all elements a, b). Then, $a\backslash b = b/a$, and one considers one residual only, denoted $a \to b$. All results from sections 2, 3 and 4 can be proved for $\mathbf{DFNL}_e$, and proofs are similar as above. Exception: not every $\epsilon-$free context-free language can be generated by a categorial grammar based on $\mathbf{DFNL}_e$.

One can add the multiplicative constant 1, interpreted as the unit in unital groupoids. We need the axiom (1R): $\Rightarrow 1$, and the rule:

$$\text{(1Ll)}\ \frac{X[Y] \Rightarrow \alpha}{X[(1,Y)] \Rightarrow \alpha},\ \text{(1Lr)}\ \frac{X[Y] \Rightarrow \alpha}{X[(Y,1)] \Rightarrow \alpha}.$$

The empty antecedent is understood as the empty structure Λ, and one admits $(\Lambda, X) = X$, $(X, \Lambda) = X$ in metatheory. Again, there are no problems with adapting our results for **DFNL** with 1 and $\mathbf{DFNL}_e$ with 1. $\overline{T}$ contains 1, for any set T. 1 is an interpolant of Λ. Models are unital groupoids. $f(\Lambda) = 1$, for any assignment f. $1_C = C(1)$ is the unit of $C[\mathcal{M}]$.

One can also add the weakening rule:

$$\text{(WEA)}\ \frac{X[\Lambda] \Rightarrow \alpha}{X[Y] \Rightarrow \alpha}.$$

Models of the resulting system are integral lattice-ordered residuated groupoids (1 is the upper bound). Applying methods of this paper, one can find new proofs of some results of [5].

Additive constants $\bot$ and $\top$ can also be added, with axioms:

$$(\bot\text{L})\ X[\bot] \Rightarrow \alpha\,,\ (\top\text{R})\ X \Rightarrow \top\,.$$

They are interpreted as the lower bound and the upper bound, respectively, of the lattice. $\overline{T}$ must contain these constants. In the proof of (an analogue of) Lemma 1, one must consider new cases: $X[Y] \Rightarrow \alpha$ is an axiom ($\bot$L) or ($\top$R). For the first case, if Y contains $\bot$ then $\bot$ is an interpolant of Y; otherwise, $\top$ is an interpolant of Y. For the second case, $\top$ is an interpolant of Y. In the proof of Lemma 3, $M(\overline{T}, \Phi)$ interprets $\bot$ as $C_{\overline{T}}(\emptyset)$ and $\top$ as $\overline{T}^*$.

Instead of one binary product $\cdot$ one may admit a finite number of operations $o, o', \ldots$ of arbitrary arity: nullary, unary, binary, ternary and so on. Each $n-$ary operation o is associated with n residual operations o^i, for $i = 1, \ldots, n$ (nullary operations have no residuals). In models, one assumes the (generalized) residuation law:

$$o(a_1, \ldots, a_n) \leq b \text{ iff } a_i \leq o^i(a_1, \ldots, b, \ldots, a_n)\,, \tag{9}$$

for all $i = 1, \ldots, n$ (on the right-hand side, b is the $i-$th argument of o^i). The corresponding formal system, called Generalized Lambek Calculus **GLC**, was presented in [11]. To each $n-$ary operation o one attributes a structure constructor $(X_1, \ldots, X_n)_o$, and formula structures can contain different structure constructors. Unary operations can be identified with (different) unary modalities. **GLC** represents a multi-modal variant of **NL**.

$$(o\mathrm{L})\ \frac{X[(\alpha_1, \ldots, \alpha_n)_o] \Rightarrow \gamma}{X[o(\alpha_1, \ldots, \alpha_n)] \Rightarrow \gamma},$$

$$(o\mathrm{R})\ \frac{X_1 \Rightarrow \alpha_1; \ldots; X_n \Rightarrow \alpha_n}{(X_1, \ldots, X_n)_o \Rightarrow o(\alpha_1, \ldots, \alpha_n)},$$

$$(o/i\mathrm{L})\ \frac{X[\alpha_i] \Rightarrow \gamma; Y_1 \Rightarrow \alpha_1; \ldots; Y_n \Rightarrow \alpha_n}{X[(Y_1, \ldots, (o/i)(\alpha_1, \ldots, \alpha_n), \ldots, Y_n)_o] \Rightarrow \gamma},$$

$$(o/i\mathrm{R})\ \frac{(\alpha_1, \ldots, X, \ldots, \alpha_n)_o \Rightarrow \alpha_i}{X \Rightarrow (o/i)(\alpha_1, \ldots, \alpha_n)}.$$

The system contains (CUT). Rules (oL), (oR) are also admitted for nullary operation symbols o. Rules ((o/iL), (o/iR) are admitted for non-nullary operation symbols only. In (o/iL), the sequent $Y_i \Rightarrow \alpha_i$ does not appear among premises. In the premise of (o/iR), X is the $i-$th argument of $(-, \ldots, -)_o$.

The consequence relation of **GLC** is polytime, and the corresponding categorial grammars generate $\epsilon-$free context-free languages [11]. All results of this paper can easily be adapted for **GLC** with $\wedge, \vee$ and distribution (and, possibly, (EXC) for some binary operations, multiplicative units for them, and $\bot, \top$). In powerset algebras, one defines $o(U_1, \ldots, U_n)$ as the set of all elements $o(a_1, \ldots, a_n)$ such that $a_i \in U_i$, for $i = 1, \ldots, n$. (C4) takes the form: $o(C(U_1), \ldots, C(U_n)) \subseteq C(o(U_1, \ldots, U_n))$. Each operation o in the powerset algebra induces an operation on closed sets: $o_C(U_1, \ldots, U_n) = C(o(U_1, \ldots, U_n))$.

At the end, we consider **DFNL** with $\bot, \top$ and negation $\neg$, satisfying the laws of boolean algebra. To axioms for $\bot$ and $\top$ we add:

$$(\neg 1)\ \alpha \wedge \neg\alpha \Rightarrow \bot\,,\ (\neg 2)\ \top \Rightarrow \alpha \vee \neg\alpha\,.$$

Clearly the resulting system, denoted **BFNL**, is strongly complete with respect to boolean-ordered residuated groupoids. It provides a solution of the problem of axiomatizing Lambek calculus with classical negation, discussed in [9]; a non-classical negation in Lambek calculus has been studied in [30].

The consequence relation for this system is decidable, and the corresponding categorial grammars generate $\epsilon-$free context-free languages. The proofs are similar to those for **DFNL**. One assumes that $\overline{T}$ is also closed under $\neg$. The only serious problem is to interpret $\neg$ as an operation on closed sets. This can be solved as follows. As in **GLC**, we add the product & with residual $\rightarrow$ and define $\neg\alpha = \alpha \rightarrow \bot$. We also admit the above axioms for $\neg$. It is easy to show that the latter system is conservative over the former system (also for consequence relations); every model of the former system can be expanded to a model of the latter system by interpreting & as $\wedge$ and $\alpha \rightarrow \beta$ as $\neg\alpha \vee \beta$. For the latter system, we can proceed as for **GLC** with additives.

A class of algebras possesses Finite Embeddability Property (FEP), if every finite partial subalgebra of an algebra from this class can be embedded into some finite algebra from this class [5, 14]. If the class is closed under products, then FEP is equivalent to SFMP: if a Horn formula is not valid in this class, then it can be falsified in some finite algebra from this class. In our setting, Horn formulas are represented by deduction patterns $\Phi \vdash X \Rightarrow \alpha$. As a consequence of Lemma 3 for **BFNL**, we obtain:

Theorem 2. **BFNL** *possesses SFMP. The class of boolean-ordered residuated groupoids possesses FEP.*

We can also add to **DFNL** with $\bot, \top$ new axioms $\alpha\&\beta \Rightarrow \alpha \wedge \beta$, $\alpha \wedge \beta \Rightarrow \alpha\&\beta$. Then, $(\neg 1)$ is provable, but $(\neg 2)$ is not. The resulting logic of $\wedge, \vee, \rightarrow, \bot, \top$ is intuitionistic, and the corresponding models are Heyting algebras with an additional residuated structure. We denote this system by **HFNL**. Its models are called Heyting-ordered residuated groupoids. All results of this paper can be obtained for **HFNL**.

Theorem 3. **HFNL** *possesses SFMP. The class of Heyting-ordered residuated groupoids possesses FEP.*

Clearly, these theorems can be generalized to multi-modal systems. An interesting special case will be classical modal logics, corresponding to **GLC** with booleans, restricted to unary operations o. Our methods yield FEP of boolean algebras with operators. Caution: if o is denoted by $\Diamond$, then $o/1$ is not the classical $\Box$, defined as $\Box\alpha = \neg\Diamond\neg\alpha$. o and $o/1$ form an adjoint pair, and either gives rise to a different pair of classical modalities, satisfying all laws of modal logic **K**. Now, α is a theorem iff $\top \Rightarrow \alpha$ is provable in the sequent system.

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